3.2 The rank of a matrix and matrix inverse

（這部份內容與課本有些不同！請注意^^）

Def: let $A = \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} = [C_1, \cdots, C_n] \in M_{m \times n}$.

$$\text{row space}(A) \overset{\text{def}}{=\!=} \operatorname{span}\{R_1, \cdots, R_m\}$$

$$\text{column space}(A) \overset{\text{def}}{=\!=} \operatorname{span}\{C_1, \cdots, C_n\}$$

$$\text{row rank}(A) \overset{\text{def}}{=\!=} \dim(\text{row space}(A))$$

$$\text{Column rank}(A) \overset{\text{def}}{=\!=} \dim(\text{column space}(A))$$

Fact: let $A \in M_{m \times n}(F)$

Then $\text{column space}(A) = \{Ax : x \in F^n\}$

pf: let $A = [C_1, \cdots, C_n]$.

Then $\{Ax : x \in F^n\}$

$$= \left\{ \sum_{i=1}^{n} x_i C_i : x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in F^n \right\}$$

$$= \operatorname{span}\{C_1, \cdots, C_n\} = \text{column space}(A).$$

Lemma* Suppose $A \in M_{m\times n}(F)$. Then we have **column rank (A) = rank (L_A)**, where $L_A \in \mathcal{L}(F^n, F^m)$ s.t. $L_A(x) = Ax$ for each $x \in F^n$.

pf: $\text{rank}(L_A) = \dim(R(L_A)) = \dim\{L_A(x) : x \in F^n\}$
$= \dim\{Ax : x \in F^n\} = \dim(\text{column space}(A))$
$= \text{column rank}(A)$.

Lemma A $A \xrightarrow{\text{ero}} B \Rightarrow \text{row rank}(A) = \text{row rank}(B)$

elementary row operations

pf: It suffices to show row space (A) = row space (B)!

Lemma B $A \xrightarrow{\text{ero}} B \Rightarrow \{x \in F^n : Ax = 0\} = \{x \in F^n : Bx = 0\}$

$A \in M_{m\times n}(F)$; zero vector of F^m

pf: $A \xrightarrow{\text{ero}} B \Rightarrow B = EA$ for some elementary matrix $E \in M_{m\times m}$

$\Rightarrow \{x \in F^n : Ax = 0\}$
$= \{x \in F^n : EAx = 0\}$ ($\because E^{-1}$ exists)
$= \{x \in F^n : Bx = 0\}$

Reduced Row Echelon Form

Def: A matrix is said to be in reduced row echelon form (or row-reduced echelon form) if it satisfies

ⓐ Any row containing a nonzero entry precedes any row in which all the entries are zero (if any).

ⓑ The first nonzero entry in each row is the only nonzero entry in its column.

ⓒ The first nonzero entry in each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row.

$$\begin{bmatrix} 1 & 0 & * & 0 & * & * \\ 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thm 3.14 P187 Every matrix can be transformed into reduced row echelon form by a sequence of elementary row operations.

(Gaussian elimination)

Ex: let $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}$.

① transforms A into a reduced row echelon form by a sequence of eros.

② Find row rank (A).

Sol:

① $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{pmatrix} \xrightarrow{\text{ero}} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix}$

$\xrightarrow{\text{ero}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ ← Answer

③ Note that

row rank(A) = row rank $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 2$,

since $\{(1,0,1), (0,1,1)\}$ is a basis for row space (A).

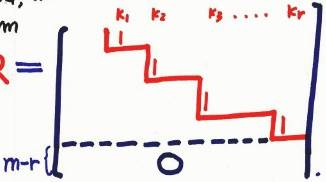
Thm * Suppose $A \in M_{m\times n}(F)$.

If row rank$(A) = r$ then we have
$$\dim\{x \in F^n : Ax = 0\} = n - r.$$

Pf: Thm 3.14 [P187] said that $\exists$ a matrix R of reduced row echelon form s.t.
$$A \xrightarrow{\text{ero's}} R$$

let row-rank$(A) = r$, and hence row-rank$(R) = r$.

WLOG, we assume that R has the following form

$R =$ 

Let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $J = \{1, 2, \cdots, n\} - \{k_1, k_2, \cdots, k_r\}$.

The equations $Rx = 0$ (zero vector in F^n) has the following form:

pf (continued)

$$x_{k_1} + \sum_{j\in J} c_{1j} x_j = 0$$
$$x_{k_2} + \sum_{j\in J} c_{2j} x_j = 0$$
$$\vdots \qquad \vdots$$
$$x_{k_r} + \sum_{j\in J} c_{rj} x_j = 0.$$

Therefore we have

$$\dim\{x\in F^n : Ax=0\}$$
$$= \dim\{x\in F^n : Rx=0\}$$ why? see Thm 3.1
$$= n-r$$ why?

QED

Example for Thm*

Assume that row-rank $(A)=2$, $A\in M_{3\times 3}$

and $A \xrightarrow{\text{ero}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \overset{\text{def}}{=\!=} R$

Then $\dim\{x\in F^3 : Ax=0\}$

$= \dim\{x\in F^3 : Rx=0\}$

$= \dim\left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{matrix} x_1+x_3=0 \\ x_2+x_3=0 \end{matrix}\right\}$

$= \dim\left\{\begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} : x_3\in\mathbb{R}\right\} = 1 = \underset{\substack{\| \\ 3}}{n} - \underset{\substack{\| \\ 2}}{r}$

Thm For any matrix $A \in M_{m\times n}$, we have

row rank (A) = column rank (A)

pf: Column rank (A)

$= \dim(\text{column space}(A))$

$= \dim\{Ax : x \in F^n\}$

$= \text{rank}(L_A)$ note: $L_A : F^n \to F^m$; $L_A(x) = Ax$

$= n - \text{nullity}(L_A)$

$= n - \dim\{x \in F^n : Ax = 0\}$

$= n - (n - \text{row rank}(A))$ ($\because$ Thm $*$)

$= \text{row rank}(A)$

QED

Def: $\text{rank}(A) \overset{\text{def}}{=} \text{row rank}(A)$

Thm 3.7 P159 $\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$

pf: Let $A \in M_{m\times t}$, $B \in M_{t\times n}$.

Let $A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_t \end{bmatrix}$

pf (continued)

row space(AB)

$$= \text{row space}\left(\begin{bmatrix} A_1 B \\ A_2 B \\ \vdots \\ A_m B \end{bmatrix}\right)$$

$$= \text{row space}\left(\begin{bmatrix} \sum_{j=1}^{t} a_{1j} B_j \\ \sum_{j=1}^{t} a_{2j} B_j \\ \vdots \\ \sum_{j=1}^{t} a_{mj} B_j \end{bmatrix}\right)$$

where $A_i = [a_{i1}\ a_{i2} \cdots a_{it}]$

$\subseteq$ rowspace(B). and hence $\text{rank}(AB) \leq \text{rank}(B)$

Similarly one can show

column space$(AB) \subseteq$ column spae(A),

and hence $\text{rank}(AB) \leq \text{rank}(A)$.

Thm 3.5 P153 The rank of any matrix equals to the maximum number of its linearly independent columns (rows).

Fact: P158 $\text{rank}(A^t) = \text{rank}(A)$.

Thm If $A \in M_{n\times n}$ is invertible then $\text{rank}(A)=n$.

pf: A is invertible

$\Rightarrow \exists\, B \in M_{n\times n}$ s.t. $AB=I_n$

$\Rightarrow n=\text{rank}(I_n)=\text{rank}(AB)\leq \text{rank}(A)$

$\Rightarrow \text{rank}(A)=n$

Corollary 3 P159 If $A \in M_{n\times n}$ has $\text{rank}(A)=n$

then ① A is invertible.

② A is a product of elementary matrices.

pf: Note that $\exists$ a matrix $R \in M_{n\times n}$ which has reduced row echelon form s.t.

$$A \xrightarrow{\text{ero's}} R$$

$\text{rank}(A)=n \Rightarrow \text{rank}(R)=n \Rightarrow R=I_n$

We also note that $R=E_1E_2\cdots E_\ell A$ for some elementary matrices $E_1, E_2, \cdots, E_\ell$.

Therefore $A=E_\ell^{-1}E_{\ell-1}^{-1}\cdots E_2^{-1}E_1^{-1}$.

Note that the inverse of an elementary matrix is also an elementary matrix.

Thm 3.4 P153 Let $A \in M_{m\times n}$. If $P \in M_{m\times m}$, $Q \in M_{n\times n}$ are invertible matrices, then

① $\mathrm{rank}(AQ) = \mathrm{rank}(A)$

② $\mathrm{rank}(PA) = \mathrm{rank}(A)$

③ $\mathrm{rank}(PAQ) = \mathrm{rank}(A)$

pf: It follows from the fact that an invertible matrix is a product of elementary matrices.

Remark: See Examples 1, 2 of P154.
Example 3 of P155
Example 4 of P160

Thm Let $A \in M_{n\times n}$. If $BA = I_n$, where $B \in M_{n\times n}$ then $AB = I_n$.

pf: $\mathrm{rank}(BA) \leq \mathrm{rank}(A)$

$\Rightarrow n \leq \mathrm{rank}(A)$

$\Rightarrow \mathrm{rank}(A) = n$

$\Rightarrow A$ is invertible ($\because$ Corollary 3 P159)

$\Rightarrow \exists\, C \in M_{n\times n}$ s.t. $AC = I_n$

$\Rightarrow C = B$ why?

QED

Thm 3.6 P155 $A \in M_{m\times n}$, $\mathrm{rank}(A) = r$. Then

(1) $r \le m$, $r \le n$

(2) $A \xrightarrow{\text{a sequence of ero's \& eco's}} \begin{array}{cc} \left(\begin{array}{c:c} I_r & O \\ \hdashline O & O \end{array}\right) & \begin{array}{l} r \\ m-r \end{array} \\ \begin{array}{cc} r & n-r \end{array} & \end{array}$

pf: (2) Thm 3.14 P187 says that

$\exists$ a matrix R in reduced row echelon form

such that

$A \xrightarrow{\text{ero's}} R$ say $R = \begin{bmatrix} 1 & 0 & * & 0 & * & * \\ 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} r \\ \left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\} m-r \end{array}$

Clearly $R \xrightarrow{\text{eco's}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{array}{cc} \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} & \begin{array}{l} \}r \\ m-r \end{array} \\ \begin{array}{cc} r & n-r \end{array} & \end{array}$

QED

Def: Let $A \in M_{m\times n}$, $B \in M_{m\times p}$.

The augmented matrix $(A|B) \overset{\text{def}}{=} (A\ B) \in M_{m\times(n+p)}$

Fact: Let $A \in M_{n\times n}$. Then

$\mathrm{rank}(A) = n \Leftrightarrow A$ is invertible

$\Leftrightarrow A$ is a product of elementary matrices

$\Leftrightarrow A \xrightarrow{\text{a sequence of eco's \& ero's}} I_n$

How to find A^{-1} (if exists of course)

Ex5 P162 $A = \begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{pmatrix}$

$\exists$ a matrix $R \in M_{3\times 3}$ in reduced row echelon form

s.t. $A \xrightarrow{\text{ero's}} R$.

i.e. $\exists$ elementary matrices $E_1, E_2, \cdots, E_\ell$ s.t.

$$E_\ell E_{\ell-1} E_{\ell-2} \cdots E_1 A = R,$$ and hence

If A is invertible then $A^{-1} = E_\ell E_{\ell-1} \cdots E_1$.

$$\left(\begin{array}{ccc|ccc} 0 & 2 & 4 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{ero}} \left(\begin{array}{ccc|ccc} 2 & 4 & 2 & 0 & 1 & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\text{ero}} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\vdots$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{8} & -\frac{5}{8} & \frac{3}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{array}\right)$$

How to find T^{-1} (if exists, of course)

Ex7 [164] $T \in \mathcal{L}(P_2(\mathbb{R}), P_2(\mathbb{R}))$ s.t.

$$T(f(x)) = f(x) + f'(x) + f''(x)$$

(1) Is T invertible ?

(2) If T is invertible, then compute T^{-1}

Sol: Let β be the standard ordered basis for $P_2(\mathbb{R})$. Note that Thm 2.18 [P101] said that

T is invertible $\Leftrightarrow$ $[T]_\beta$ is invertible

$$[T]_\beta = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

One Way to see if $[T]_\beta$ is invertible is to compute

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

So $([T]_\beta)^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ and T is invertible !

Note that

$$[T^{-1}]_\beta = ([T]_\beta)^{-1}. \quad (\because \text{Thm 2.18 } ^{P101})$$

Sol (continued)

We also note that

$$[T^{-1}(a+bx+cx^2)]_\beta$$
$$= [T^{-1}]_\beta^\beta [a+bx+cx^2]_\beta$$
$$= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a-b \\ b-2c \\ c \end{pmatrix}.$$

Therefore

$$T^{-1}(a+bx+cx^2) = (a-b) + (b-2c)x + cx^2$$

Why?

∎