3.3 System of linear equations – Theoretical concepts

Recall: Let $A = (a_{ij}) \in M_{m\times n}(F)$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$. Then $Ax = b$ is called a system of m linear equations in n unkowns over the field F.

Def: $Ax = b$ is called consistent if there is a solution $\hat{x}$ s.t. $A\hat{x} = b$; otherwise it is called inconsistent.

defined in the recall

Thm 3.8 P171 Let $K = \{x \in F^n : Ax = 0\}$.

Then ① $K = N(L_A)$.

② K is a subspace of F^n

③ $\dim K = n - \operatorname{rank}(A)$

pf: ③
$$\begin{aligned}\dim K &= \dim N(L_A) \\ &= \operatorname{nullity}(L_A) \quad (\because L_A : F^n \to F^m \text{ and dimension theorem}) \\ &= \dim(F^n) - \operatorname{rank}(L_A) \\ &= n - \operatorname{rank}(A) \quad \text{why? column rank}(A) = \operatorname{rank}(L_A)\end{aligned}$$

Corollary p171 If $m<n$, then $Ax=0$ has a nonzero solution.

pf: $\dim\{x\in F^n : Ax=0\}$

$$= n-\text{rank}(A) \geq n-\min\{m,n\} = n-m>0$$

So $\exists\, x\neq 0$ s.t. $Ax=0$. **QED**

Thm 3.9 p172 let $K=\{x\in F^n : \overset{M_{m\times n}}{A}x=\overset{F^m}{b}\}$ and $K_H=\{x\in F^n : Ax=0\}$. If $s\in K$ then $K=\{s+v : v\in K_H\}$.

Thm 3.10 p174 let $A\in M_{n\times n}$, $b\in M_{n\times 1}$. Then

$Ax=b$ has exactly one solution

$\Leftrightarrow$ A is invertible.

Thm 3.11 p174 let $A\in M_{m\times n}$, $b\in M_{m\times 1}$.

Then $Ax=b$ has a solution $\Leftrightarrow$ $\text{rank}(A) = \text{rank}(A|b)$

pf: "$\Leftrightarrow$" $Ax=b$ has a solution

$\Leftrightarrow$ $b\in$ column space (A) $\Leftrightarrow$ $\text{rank}(A) = \text{rank}(A|b)$

QED

Ex6 P175 Let $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ s.t.

$$T(a,b,c) = (a+b+c,\ a-b+c,\ a+c)$$

Is $(3,3,2) \in R(T)$?

Sol: Method 1: Try to solve

$$\begin{cases} a+b+c=3 \\ a-b+c=3 \\ a \quad\ +c=2 \end{cases}$$

Method 2: Thm 3.11 P174 suggests that it suffices to consider

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \overset{?}{=} \text{rank}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 3 \\ 1 & 0 & 1 & 2 \end{array}\right)$$

Remark: in fact, Methods 1 & 2 are the same method!

why?