3.4 System of Linear Equations — computational aspects

Def: P182 Two systems of linear equations are called equivalent if they have the same solution set.

Thm 3.13 P182 Let $A \in M_{m\times n}$, $b \in M_{m\times 1}$, and let C be an invertible $m\times m$ matrix. Then the two systems $(CA)x = Cb$, $Ax = b$ are equivalent.

Corollary P182 Let $A \in M_{m\times n}$ and $b \in M_{m\times 1}$. If $(A'|b')$ is obtained from $(A|b)$ by a finite number of ero's, then $A'x = b'$ is equivalent to $Ax = b$.

Note: please check the procedure described on pages 183-185, such procedure is called Gaussian elimination which transforms a matrix into its reduced row echelon form.

How to solve $Ax=b$?

- let $A \in M_{m\times n}$, $b \in M_{m\times 1}$.

$$(A|b) \xrightarrow[\text{ero's}]{\text{Gaussian elimination}} (A'|b')$$

augmented matrix

reduced row echelon form of A'

Note that

$$\{x : Ax=b\} = \{x : A'x=b'\}.$$

Ex P187 Solve $Ax=b$, where

$$A=\begin{pmatrix} 2 & 3 & 1 & 4 & -9 \\ 1 & 1 & 1 & 1 & -3 \\ 1 & 1 & 1 & 2 & -5 \\ 2 & 2 & 2 & 3 & -8 \end{pmatrix}, \quad b=\begin{pmatrix} 17 \\ 6 \\ 8 \\ 14 \end{pmatrix}.$$

Sol:

$$(A|b) \xrightarrow[\text{ero's}]{\text{Gaussian elimination}} \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

A' $\quad$ b'

the augmented matrix of $Ax=b$

Note that $\{x : Ax=b\} = \{x : A'x=b'\}$

$$= \left\{ x : \begin{array}{l} x_1 + 2x_3 - 2x_5 = 3 \\ x_2 - x_3 + x_5 = 1 \\ x_4 - 2x_5 = 2 \end{array} \right\} = \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} : x_3, x_5 \in \mathbb{R} \right\}$$

Sol: (continued)

Note that

$\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$ is a basis for the corresponding homogeneous system of $Ax = b$.

Thm 3.15 P189 see the example in the previous page!

Thm 3.16 Let $A \in M_{m\times n}$ (say $A = [A_1, \cdots, A_n]$), $\operatorname{rank}(A) = r > 0$, and let B be the reduced row echelon form of A. Then

ⓐ The # of nonzero rows in B is r.

ⓑ For each $i = 1, 2, \cdots, r$, there is a column b_{j_i} of B s.t. $b_{j_i} = e_i$.

note: 課本此處描述待
∵ $b_{j_i} \in M_{m\times 1}$ 但 e_i 為 I_r 的第 i column.
故 e_i 下面還要補 $m-r$ 個 0's.

ⓒ The columns of A numbered $j_1, j_2, \cdots, j_r$ are linearly independent.

ⓓ For each $k = 1, 2, \ldots, n$ if column k of B is $d_1 e_1 + d_2 e_2 + \cdots + d_r e_r$ then column k of A is $d_1 A_{j_1} + d_2 A_{j_2} + \cdots + d_r A_{j_r}$

pf: "(a) ⇔ (b)"

Note that $\exists$ invertible matrix $M \in M_{m\times m}$

s.t. $B = MA$

where B has the form

$$B = \begin{array}{c} \begin{array}{cccc} B_{j_1} & B_{j_2} & B_{j_3} \cdots & B_{j_r} \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & & \\ 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \end{array}\right] \end{array}$$

(c) let $A = [A_1, A_2, \cdots, A_n]$

Suppose $c_1 A_{j_1} + c_2 A_{j_2} + \cdots + c_r A_{j_r} = 0$ in $M_{m\times 1}$

We have $c_1 MA_{j_1} + c_2 MA_{j_2} + \cdots + c_r MA_{j_r} = 0$ in $M_{m\times 1}$

i.e. $c_1 B_{j_1} + c_2 B_{j_2} + \cdots + c_r B_{j_r} = 0$

and hence $c_1 = c_2 = \cdots = c_r = 0$

Pf (continued)

(d) Let $B = [B_1, B_2, \cdots, B_n]$

Suppose

$$B_k = d_1 e_1 + d_2 e_2 + \cdots + d_r e_r$$

Then we have

$$B_k = d_1 B_{j_1} + d_2 B_{j_2} + \cdots + d_r B_{j_r}$$

$$= d_1 MA_{j_1} + d_2 MA_{j_2} + \cdots + d_r MA_{j_r}$$

$$= M(d_1 A_{j_1} + d_2 A_{j_2} + \cdots + d_r A_{j_r})$$

So $M^{-1} B_k = d_1 A_{j_1} + d_2 A_{j_2} + \cdots + d_r A_{j_r}$

Therefore $A_k = d_1 A_{j_1} + d_2 A_{j_2} + \cdots + d_r A_{j_r}$.

QED

Corollary The reduced row echelon form of a matrix is unique.

Pf: Use Thm 3.16 (d)

pf (continued) (sketch)

let $A \xrightarrow{\text{ero's}} B$, $A \xrightarrow{\text{ero's}} B'$, where

B, B' are reduced row echelon form's of A,

say

$$B = \begin{array}{cccccc} j_1 & j_2 & j_3 & \cdots & j_r & \\ \multicolumn{5}{c}{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}} & \end{array} \overset{\text{def}}{=} [B_1, \ldots, B_n]$$

$$A \overset{\text{def}}{=} [A_1, \ldots, A_n]$$

$$B' = \begin{array}{ccccc} j_1' & j_2' & j_3' & \cdots & j_r' \\ \multicolumn{5}{c}{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}} \end{array} \overset{\text{def}}{=} [B_1', \ldots, B_n']$$

Note $j_1 < j_1' \Rightarrow B_{j_1}' = 0 \times B_{j_1'}' \underset{\uparrow \text{ Thm 3.16 (d)}}{\Rightarrow} A_{j_1} = 0 \times A_{j_1'} = 0$ a contradiction since $\{A_{j_1}\}$ is a l.i. set by Thm 3.16 (c)

Therefore $j_1 = j_1'$.

Note $j_2 < j_2' \Rightarrow B_{j_2}' = d_1 \times B_{j_1'}' + 0 \times B_{j_2'}'$

$\Rightarrow A_{j_2} \underset{\uparrow \text{ Thm 3.16(d)}}{=} d_1 A_{j_1'} \underset{\uparrow\, j_1 = j_1'}{=} d_1 A_{j_1}$ a contradiction to $\{A_{j_1}, A_{j_2}\}$ is a l.i. set by Thm 3.16 (c)

Therefore $j_2 = j_2'$.

Note $j_3 < j_3' \Rightarrow B_{j_3}' = c_1 B_{j_1'}' + c_2 B_{j_2'}' + 0 B_{j_3'}' \overset{*}{\Rightarrow} A_{j_3} = c_1 A_{j_1} + c_2 A_{j_2}$

* follows from Thm 3.16 (d) and $j_1 = j_1'$, $j_2 = j_2'$.

a contradiction to $\{A_{j_1}, A_{j_2}, A_{j_3}\}$ is a l.i. set.

Therefore $j_3 = j_3'$

QED

Ex 2 p191

$$
A = \begin{pmatrix} 2 & 4 & 6 & 2 & 4 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 4 & 8 & 0 & 0 \\ 3 & 6 & 7 & 5 & 9 \end{pmatrix} \xrightarrow{\text{ero's}} \begin{array}{c} \begin{array}{ccccc} 1 = j_1 & & 3 = j_2 & & 5 = j_3 \end{array} \\ \begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} = B
$$

B ← reduced row echelon form of A

let $B = [B_1, B_2, B_3, B_4, B_5]$

$A = [A_1, A_2, A_3, A_4, A_5]$

Note that

$$B_2 = 2B_{j_1}, \quad B_4 = 4B_{j_1} + (-1)B_{j_2}.$$

$$A_2 = 2A_{j_1}, \quad A_4 = 4A_{j_1} + (-1)A_{j_2}.$$

Ex p192 let $S=\left\{ \begin{pmatrix}2\\-3\\5\end{pmatrix}, \begin{pmatrix}8\\-12\\20\end{pmatrix}, \begin{pmatrix}1\\0\\-2\end{pmatrix}, \begin{pmatrix}0\\2\\-1\end{pmatrix}, \begin{pmatrix}7\\2\\0\end{pmatrix} \right\}$.

Find a basis for span(S).

Sol: Let $A=\begin{pmatrix}2 & 8 & 1 & 0 & 7\\ -3 & -12 & 0 & 2 & 2\\ 5 & 20 & -2 & -1 & 0\end{pmatrix}$

Note that

$$A \xrightarrow{\text{ero's}} \begin{array}{c} \begin{array}{ccccc} j_1 & & j_2 & j_3 & \end{array} \\ \begin{pmatrix}1 & 4 & 0 & 0 & 2\\ 0 & 0 & 1 & 0 & 3\\ 0 & 0 & 0 & 1 & 4\end{pmatrix} \end{array}$$

reduced row echelon form of A

So rank(A) = 3 and

$\beta=\left\{ \begin{pmatrix}2\\-3\\5\end{pmatrix}, \begin{pmatrix}1\\0\\-2\end{pmatrix}, \begin{pmatrix}0\\2\\-1\end{pmatrix} \right\}$ is a l.i. subset of S.

Moreover, β spans column space (A). ($\because$ Thm 3.16 (d))

Therefore β is a basis for span(S).

How to reduce a finite generating set to a basis

Ex3 P193 Let $S = \{ 2+x+2x^2+3x^3,\ 4+2x+4x^2+6x^3,\ 6+3x+8x^2+7x^3,\ 2+x+5x^3,\ 4+x+9x^3 \}$

Find a basis for span(S).

Sol: let

$$A = \begin{pmatrix} 2 & 4 & 6 & 2 & 4 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 4 & 8 & 0 & 0 \\ 3 & 6 & 7 & 5 & 9 \end{pmatrix}$$

Note that

$$A \xrightarrow{\text{ero's}} \begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So

$\beta = \{ 2+x+2x^2+3x^3,\ 6+3x+8x^2+7x^3,\ 4+x+9x^3 \}$

is a basis for span(S).

How to extend a l.i. subset S of a vector space V to a basis for V.

EX4 P193 Let $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{pmatrix} \in \mathbb{R}^5 : x_1 + 7x_2 + 5x_3 - 4x_4 + 2x_5 = 0 \right\}$

V is a vector space of $\mathbb{R}^5$ and

$S = \left\{ \underset{\alpha_1}{\begin{pmatrix} -2 \\ 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}}, \underset{\alpha_2}{\begin{pmatrix} 1 \\ 1 \\ -2 \\ -1 \\ -1 \end{pmatrix}}, \underset{\alpha_3}{\begin{pmatrix} -5 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}} \right\}$ is a l.i. subset of V. Please extend S to a basis for V.

Sol: For $\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} \in V$, we have

$$\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} = \begin{pmatrix} -7x_2 - 5x_3 + 4x_4 - 2x_5 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$= x_2 \underset{\beta_1}{\begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}} + x_3 \underset{\beta_2}{\begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}} + x_4 \underset{\beta_3}{\begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}} + x_5 \underset{\beta_4}{\begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}.$$

Note that $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ is a basis for V.

Sol: (continued)

Note that

$$[\alpha_1, \alpha_2, \alpha_3] \xrightarrow{\text{ero's}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Consider

$$[\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4]$$

$$\xrightarrow{\text{ero's}} \begin{array}{c} \begin{array}{ccccccc} j_1 & j_2 & j_3 & & & j_4 & \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

So S can be extended to a basis

$$\left\{ \begin{pmatrix} -2 \\ 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ for } V.$$

Note：課本的講法需稍加修正！