Determinants

Def: [p200] If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the determinant of A

det(A) or $|A| \overset{def}{=} ad - bc$.

Def: If $A = (a_{ij}) \in M_n(F)$.

The (i, j) minor submatrix $A_{ij} \overset{def}{=}$ the submatrix of A resulting from the deletion of row i and column j. note：課本用 $\tilde{A}_{ij}$ 這符号, see p209.

The (i, j) minor of $A \overset{def}{=} \det(A_{ij})$.

note: Here we assume that what is meant by their determinant is known.

The determinant of $A \underset{p209}{\overset{def}{=}} \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$

The (i, j) cofactor of $A \overset{def}{=} (-1)^{i+j} \det(A_{ij})$

Notation $\sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$ is called the cofactor expansion along the ith row of A.

p218, also p222 Ex 23.

Thm The determinant of a lower triangular matrix A is the product of its diagonal entries.

pf: let $A = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix}$

$$\det A = a_{11} \det A_{11} = a_{11}(a_{22} \cdots\cdots a_{nn})$$

by induction on n. **QED** (sketch)

Thm 4.3 p212 let $A = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}$, $B = [B_1, \cdots B_n] \in M_{n\times n}$.

Then

$$\det \begin{bmatrix} A_1 \\ \vdots \\ A_{i-1} \\ u+kv \\ A_{i+1} \\ \vdots \\ A_n \end{bmatrix} = \det \begin{bmatrix} A_1 \\ \vdots \\ A_{i-1} \\ u \\ A_{i+1} \\ \vdots \\ A_n \end{bmatrix} + k \det \begin{bmatrix} A_1 \\ \vdots \\ A_{i-1} \\ v \\ A_{i+1} \\ \vdots \\ A_n \end{bmatrix}$$ and

we also have

$$\det [B_1, \cdots, B_{j-1}, p+kq, B_{j+1}, \cdots B_n]$$

$$= \det [B_1, \cdots, B_{j-1}, p, B_{j+1}, \cdots, B_n] + k \det [B_1, \cdots, B_{j-1}, q, B_{j+1} \cdots B_n]$$

Pf (Thm 4.3, sketch)

By induction on n.

Let $u = (b_1, b_2, \cdots, b_n)$, $v = (c_1, c_2, \cdots, c_n)$

Let $A = \begin{bmatrix} A_1 \\ \vdots \\ A_{r-1} \\ u+kv \\ A_{r+1} \\ \vdots \\ A_n \end{bmatrix}$, $B = \begin{bmatrix} A_1 \\ \vdots \\ A_{r-1} \\ u \\ A_{r+1} \\ \vdots \\ A_n \end{bmatrix}$ and $C = \begin{bmatrix} A_1 \\ \vdots \\ A_{r-1} \\ v \\ A_r \\ \vdots \\ A_n \end{bmatrix}$

$r=1 \Rightarrow \det A = \det B + k \det C$ Easy!

Suppose $r \geq 2$.

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \left(\det B_{1j} + k \det C_{1j} \right) \quad \text{(induction hypothesis)}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det B_{1j} + k \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det C_{1j}$$

$$= \det B + k \det C.$$

QED

Corollary: If $A = \begin{bmatrix} A_1 \\ \vdots \\ A_{i-1} \\ 0 \\ A_{i+1} \\ \vdots \\ A_n \end{bmatrix} \in M_{n\times n}$ then $\det(A) = 0$.

pf: $\det A = \det A + \det A$ by Thm 4.3 (P212) and hence $\det A = 0$.

QED.

Lemma (p213) If $A = \begin{array}{c} \\ i \end{array}\begin{array}{c} \begin{array}{ccccccc} & & & j & & & \end{array} \\ \begin{bmatrix} & & & \vdots & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & & & & \end{bmatrix} \end{array} \in M_{n\times n}$

then $\det(A) = (-1)^{i+j}\det(A_{ij})$

Thm 4.4 (p215) If $A = (a_{ij}) \in M_{n\times n}(F)$,

then for any $1 \le i \le n$,

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

pf: Let $B_j \stackrel{def}{=}$ the matrix obtained from A by replacing row i of A by e_j.

$$\det(A) = \det\begin{bmatrix} A_1 \\ \vdots \\ A_{i-1} \\ \sum_j a_{ij} e_j \\ A_{i+1} \\ \vdots \\ A_n \end{bmatrix} \quad \text{where } A = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}$$

$$= \sum_{j=1}^{n} a_{ij} \det(B_j) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

Corollary p215 If $A=(a_{ij})=\begin{array}{c}\\ r\\ \\ s\\ \\ \end{array}\begin{pmatrix} A_1 \\ \vdots \\ u \\ \vdots \\ u \\ \vdots \\ A_n \end{pmatrix} \in M_{n\times n}$ then $\det A = 0$

pf (sketch) By induction on n.

For $i\neq r$, $i\neq s$, $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\tilde{A}_{ij})$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij}\, 0 = 0$$

↑ induction hypothesis.

QED.

Thm 4.5 p216 let $A=\begin{bmatrix} A_1 \\ \vdots \\ A_r \\ \vdots \\ A_s \\ \vdots \\ A_n \end{bmatrix}$, $B=\begin{bmatrix} A_1 \\ \vdots \\ A_s \\ \vdots \\ A_r \\ \vdots \\ A_n \end{bmatrix} \in M_{n\times n}(F)$.

i.e. B is obtained from A by interchanging two rows of A.

Then $\det(B) = -\det(A)$.

pf:

$$0=\begin{pmatrix} A_1 \\ \vdots \\ A_r+A_s \\ \vdots \\ A_r+A_s \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} A_1 \\ \vdots \\ A_r \\ \vdots \\ A_r \\ \vdots \\ A_n \end{pmatrix} + \begin{pmatrix} A_1 \\ \vdots \\ A_r \\ \vdots \\ A_s \\ \vdots \\ A_n \end{pmatrix} + \begin{pmatrix} A_1 \\ \vdots \\ A_s \\ \vdots \\ A_r \\ \vdots \\ A_n \end{pmatrix} + \begin{pmatrix} A_1 \\ \vdots \\ A_s \\ \vdots \\ A_s \\ \vdots \\ A_n \end{pmatrix}$$

done!

Thm 4.6 P216 let $A = \begin{bmatrix} A_1 \\ \vdots \\ \vdots \\ A_n \end{bmatrix}$ and $B = \begin{array}{l} \begin{bmatrix} A_1 \\ \vdots \\ A_r \\ \vdots \\ kA_r + A_s \\ \vdots \\ A_n \end{bmatrix} \end{array} \begin{array}{l} \\ \\ r \\ \\ s \\ \\ \\ \end{array} \in M_{n\times n}(F)$.

i.e. B is obtained from A by adding k times row r to row s, where $r \neq s$.

Then $\det(A) = \det(B)$.

Corollary P217 let $A \in M_{n\times n}(F)$.

If $\operatorname{rank}(A) \neq n$ then $\det(A) = 0$.

pf:

$\operatorname{rank}(A) \neq n$

$\Rightarrow \operatorname{rank}(A) < n$

$\Rightarrow$ dim rowspace $(A) < n$

$\Rightarrow$ Some row of A is a l.c. of the other rows

$\Rightarrow \det(A) = 0$ by Thm 4.3 P212, Thm 4.6 P216.

QED

Note: In face we also have $\det(A) = 0 \Rightarrow \operatorname{rank}(A) \neq n$.

Fact 1 If E is an elementary matrix (for ero 只做一次 d6) in $M_{n\times n}(F)$ then for $B \in M_{n\times n}(F)$ we have

$$\det(EB) = \det(E)\det(B)$$

pf: (sketch) For $E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$, we have

$EB = \begin{pmatrix} B_2 \\ B_1 \\ B_3 \end{pmatrix}$. Note that $\det(E) = -1$.

So
$$\det(EB) = \det\begin{pmatrix} B_2 \\ B_1 \\ B_3 \end{pmatrix} \underset{\text{Thm 4.5 P216}}{=} (-1)\det\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \det(E)\det(B)$$

QED

Fact 2. If $\operatorname{rank}(A) \neq n$ then

$$\det(AB) = \det(A)\det(B),$$

where $A, B \in M_{n\times n}(F)$.

pf: $\operatorname{rank}(A) < n$

$\Rightarrow \operatorname{rank}(AB) \leq \operatorname{rank}(A) < n$ and $\det A = 0$

$\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$

$\Rightarrow \det(AB) = 0$ by Corollary P217

$\Rightarrow \det(AB) = \det(A)\det(B)$.

QED

Fact 3 If $\text{rank}(A)=n$ then

$$\det(AB)=\det(A)\det(B),$$

where $A, B \in M_{n\times n}(F)$.

Pf: $\text{rank}(A)=n$

$\Rightarrow \exists$ elementary matrices $E_1, E_2, \dots, E_m$

s.t. $A = E_1 E_2 \cdots E_m$ P159

note: 這裡的 elementary matrix 指相關一次 ero 的矩陣！也許重新定义 elementary matrix 是更好的途径！

$$
\begin{aligned}
\Rightarrow \det(AB) &= \det(E_1E_2\cdots E_mB) \\
&\overset{\text{Fact1}}{=} \det(E_1)\det(E_2\cdots E_mB) \\
&\ \ \vdots \\
&= \det(E_1)\det(E_2)\cdots\det(E_m)\det(B) \\
&\ \ \vdots \\
&\underset{\substack{\text{Fact1} \\ \text{again.}}}{=} \det(E_1E_2\cdots E_m)\det(B) \\
&= \det(A)\det(B).
\end{aligned}
$$

QED.

Thm 4.7 For any $A, B \in M_{n\times n}(F)$,

$$\det(AB)=\det(A)\det(B).$$

Corollary p223 Let $A \in M_{n\times n}(F)$. Then

$$\operatorname{rank}(A) \neq n \iff \det(A) = 0$$

pf: (sketch) It suffices to show $\det(A)=0 \Rightarrow \operatorname{rank}(A)<n$.

Assume $\operatorname{rank}(A)=n$ i.e. A is invertible why? p159

We have $I_n = AA^{-1}$

$$\Rightarrow \det(I_n) = \det(A)\det(A^{-1})$$

$$\Rightarrow 1 = \det(A)\det(A^{-1})$$

$$\Rightarrow \det(A) \neq 0.$$

QED

Corollary p223 Let $A \in M_{n\times n}(F)$. Then

$$A \text{ is invertible} \iff \det(A) \neq 0$$

Thm 4.8 p224 For any $A \in M_{n\times n}(F)$,

$\det(A^t) = \det(A)$.

pf: (sketch) Case 1 if $\operatorname{rank}(A)<n$

Case 2 if $\operatorname{rank}(A)=n$.

QED

p231 Ex 26~27

Def: let $\underset{\substack{\shortparallel \\ (a_{ij})}}{A} \in M_{n\times n}(\mathbb{F})$

The classical adjoint of A is defined as

$$\operatorname{adj} A \overset{\text{def}}{=} \left((-1)^{i+j} \det A_{ij} \right)^t$$

Fact: let $A \in M_{n\times n}(\mathbb{F})$.

Then $A^{-1} = \dfrac{1}{\det A} \operatorname{adj} A$ provided $\det A \neq 0$.

pf:

$$A \cdot \operatorname{adj} A$$

$$= \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots\cdots & a_{nn} \end{pmatrix} \left((-1)^{j+i} \det A_{ji} \right)$$

$$= \left(\sum_{k=1}^{n} a_{ik} (-1)^{j+k} \det A_{jk} \right)$$

$$\underset{\substack{\uparrow \\ \text{why ?}}}{=} \begin{pmatrix} \det A & & 0 \\ & \det A & \\ 0 & & \ddots \\ & & & \det A \end{pmatrix} = (\det A) I_n$$

So $A^{-1} = \dfrac{1}{\det A} \operatorname{adj} A$ if $\det A \neq 0$

QED

Cramer's Rule

Thm 4.9 p224 let $A = [c_1, \cdots, c_n] \in M_{n\times n}$,

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.

n unknowns

If $Ax = b$ then we have

$$(\det A)\, x_j = \det [c_1, \cdots, c_{j-1}, b, c_{j+1}, \cdots c_n].$$

pf: let e_i be the ith column of identity matrix $I_n \in M_{n\times n}$.

let $B = [e_1, \cdots, e_{j-1}, x, e_{j+1}, \cdots, e_n]$.

Then $AB = [Ae_1, \cdots, Ae_{j-1}, Ax, Ae_{j+1}, \cdots, Ae_n]$

$$= \underbrace{[c_1, \cdots, c_{j-1}, b, c_{j+1}, \cdots, c_n]}_{\star}$$

Thus $(\det A)(\det B) = \det(\star)$

$\Rightarrow (\det A)\, x_j = \det(\star)$

QED

Ex1 p225 Use Cramer's rule to solve

$Ax=b$, where

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \text{ and } b=\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

Sol: Note that $\det A = 6 \neq 0$,

Cramer's rule applies!

Cramer's rule said that $\exists!$ solution

$x=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ s.t.

$$x_1 = \frac{\det\begin{pmatrix} 2 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}}{\det A} = \frac{5}{2}$$

$$x_2 = \frac{\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 1 & -1 \end{pmatrix}}{\det A} = -1$$

$$x_3 = \frac{\det\begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}}{\det A} = \frac{1}{2}.$$

END.

Interpre det A geometrically

Ex2 p226 A parallelepiped has the vectors

$U_1 = (1,-2,1)$, $U_2 = (1,0,-1)$ and $U_3 = (1,1,1)$

as adjacent sides. Find the volume of it.

Sol:

$$\left| \det\begin{pmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \right| = 6$$

EDN.

Ex p202 (area of a Parallelogram)

The area of the parallelogram determined by $u = (-1,5)$, $v = (4,2)$ is

$$\left| \det\begin{pmatrix} -1 & 5 \\ 4 & 2 \end{pmatrix} \right| = 18.$$