5.1 Eigenvalue & Eigenvector

Def: $T \in \mathcal{L}(V,V)$, $\dim V < \infty$

- T is called diagonalizable if $\exists$ an ordered basis β for V s.t. $[T]_\beta$ is a diagonal matrix.
- A square matrix A is called diagonalizable if L_A is diagonalizable.

Recall P92 If $A \in M_{n\times n}$ then $L_A : F^n \to F^n$ s.t. $L_A(x) = Ax$.

Remark: A square matrix A is diagonalizable if $\exists$ square matrix P s.t. $P^{-1}AP$ is a diagonal matrix.

Note: 請証明上面两个 有関 "square matrix is diagonalizable" 的定义是相同的!

Def: $T \in \mathcal{L}(V,V)$, $\dim V < \infty$ and V is a vector space over F. $A \in M_{n\times n}(F)$.

- $v \in V \setminus \{0\}$ is an eigenvector of T if $\exists\ \lambda \in F$ s.t. $T(v) = \lambda v$. Such λ is called the eigenvalue corresponding to the eigenvector v.

Def (continued)

- $v \in F^n \setminus \{0\}$ is an eigenvector of A if $Av = \lambda v$ for some $\lambda \in F$. Such λ is called the eigenvalue of A corresponding to the eigenvector v.

EX1 p247 Let $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$. Is A diagonalizable!

Sol: Consider $u_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Let $\beta = \{u_1, u_2\}$.

Then $[L_A]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$.

EX2: 3 is an eigenvalue of $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ corresponding to the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Def p248 Let $A \in M_{n\times n}(F)$.

the characteristic polynomial of $A \overset{\text{def}}{=\!=} \det(A - xI_n)$

(in terms of x)

c.p. in short

Remark: 本課程用 $P_A(x)$ 來表示矩陣 A 的 characteristic polynomial.

Thm 5.2 Let $A \in M_{n\times n}(F)$.

$\lambda \in F$ is an eigenvalue of $A \Leftrightarrow P_A(\lambda)=0$

pf: ($\Rightarrow$) λ is an eigenvalue of A

$\Rightarrow \exists\, v \in F^n \setminus \{0\}$ s.t. $Av=\lambda v$

$\Rightarrow (A-\lambda I_n)v=0$

$\Rightarrow \det(A-\lambda I_n)=0$ ($\because$ v is nontrivial)

$\Rightarrow P_A(\lambda)=0$

($\Leftarrow$) $P_A(\lambda)=0$

$\Rightarrow \det(A-\lambda I_n)=0$

$\Rightarrow \operatorname{rank}(A-\lambda I_n)<n$

$\Rightarrow \exists$ nontrivial $v\in F^n$ s.t. $(A-\lambda I_n)v=0$

why?

reason1: Consider columns of $A-\lambda I_n$.

reason2: $\operatorname{rank}(A-\lambda I_n)+\operatorname{nullity}(A-\lambda_n)=n$.

$\Rightarrow \lambda$ is an eigenvalue of A.

QED

Ex: Find the eigenvalues of $A=\begin{pmatrix}1&1\\4&1\end{pmatrix}$.

Sol: $P_A(x)=\det\begin{pmatrix}1-x&1\\4&1-x\end{pmatrix}=(x-3)(x+1)$.

$eu(A)=\{3,-1\}$

Def: The characteristic polynomial $P_T(x)$ of T is defined to be the characteristic polynomial of $[T]_\beta$ for any ordered basis β of V, here $\dim V < \infty$.

Remark: This def. is well defined.

$$[T]_\beta = [ITI]_\beta$$
$$= [I]_{\beta'}^{\beta} [T]_{\beta'}^{\beta'} [I]_{\beta}^{\beta'}$$
$$= P^{-1} [T]_{\beta'} P$$

Ex5 [P249] let $T \in \mathcal{L}(P_2(\mathbb{R}), P_2(\mathbb{R}))$ s.t. $T(f(x)) = f(x) + (x+1)f'(x)$. Let β be the standard ordered basis for $P_2(\mathbb{R})$.

(1) Find $[T]_\beta$.

(2) Find $ev(T)$

↑

eigenvalues of T.

Sol: $[T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$. $\det([T]_\beta - xI_3) = (1-x)(2-x)(3-x)$

$ev(T) = \{1, 2, 3\}$

How to find the eigenvalues of a $T \in L(V)$.

Fact: Suppose $T \in L(V)$, and V has an ordered basis β. Then

$$\lambda \in \underbrace{ev(T)}_{\text{eigenvalues of } T} \Longleftrightarrow \lambda \in ev([T]_\beta)$$

pf: $\bullet$ $Tv = \lambda v \Rightarrow [Tv]_\beta = [\lambda v]_\beta$

$$\Rightarrow [T]_\beta [v]_\beta = \lambda [v]_\beta$$

$\bullet$ $v \neq 0 \Leftrightarrow [v]_\beta \neq 0$

QED.

Fact: Suppose $T \in L(V)$, and V has an ordered basis β. Then

$$\lambda \in ev(T) \Longleftrightarrow \underbrace{P_T(\lambda)}_{\text{Characteristic polynomial of } T} = 0$$

Thm 5.3 P249 Let $A \in M_{n\times n}(F)$. Then

① $\deg(P_A(x)) = n$, and leading coefficient of $P_A(x)$ is $(-1)^n$

② A has at most n distinct eigenvalues.

Ex 6 P250

(1) Find all eigenvectors of $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.

(2) Is A diagonalizable?

Sol:

(1) $ev(A) = \{3, -1\}$.

Let $E_3 = \{ x \in \mathbb{R}^2 \setminus \{0\} : Ax = 3x \}$

$E_{-1} = \{ x \in \mathbb{R}^2 \setminus \{0\} : Ax = -x \}$

$$(A-3I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -2x_1 + x_2 = 0 \\ 4x_1 - 2x_2 = 0 \end{cases} \Rightarrow x_2 = 2x_1$$

So $E_3 = \left\{ \begin{pmatrix} t \\ 2t \end{pmatrix} : t \in \mathbb{R} \setminus \{0\} \right\}$

In a similary way, we have

$$(A+I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2x_1 + x_2 = 0 \\ 4x_1 + 2x_2 = 0 \end{cases} \Rightarrow x_2 = -2x_1$$

$E_{-1} = \left\{ \begin{pmatrix} t \\ -2t \end{pmatrix} : t \in \mathbb{R} \setminus \{0\} \right\}$

(2) $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$ consisting of eigenvalues of A $\Longrightarrow$ A is diagonalizable.

Is A diagonalizable?

Thm 5.5 P261 $A \in M_{n\times n}$.

A has n distinct eigenvalues $\Rightarrow$ A is diagonalizable

pf: Suppose $eu(A) = \{\lambda_1, \dots, \lambda_n\}$.

let $Ax_i = \lambda_i x_i$, $x_i \neq 0$.

Claim $x_1, x_2, \dots, x_n$ are l.i.

pf: By induction on n.
(sketch)

Suppose $x_1, \dots, x_t$ are l.i.

Assume $x_{t+1} = \sum_{i=1}^{t} a_i x_i$ ——— (a)

then $Ax_{t+1} = A \sum_{i=1}^{t} a_i x_i$, and hence

$$\lambda_{t+1} x_{t+1} = \sum_{i=1}^{t} a_i \lambda_i x_i \text{ ——— (b)}$$

(b) $- \lambda_{t+1}$(a) $\Rightarrow \sum_{i=1}^{t} a_i (\lambda_i - \lambda_{t+1}) x_i = 0$

$\Rightarrow a_i(\lambda_i - \lambda_{t+1}) = 0 \quad i = 1, 2, \dots, t$

$\Rightarrow a_1 = a_2 = \cdots = a_t = 0$

$\Rightarrow x_{t+1} = 0$ a contradiction.

QED of Claim

Let $P = [x_1, \dots, x_n]$. we have

$$AP = [\lambda_1 x_1, \dots, \lambda_n x_n] = P\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$ **QED.**

Example: Is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ diagonalizable?

sol: let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

$eu(A) = \{0, 2\}$.

Fact: Let $A \in M_{n\times n}(F)$. If

① $\beta = \{y_1, y_2, \dots, y_n\}$ is a basis for F^n

② $\beta \subseteq ev(A)$.

Then $Q^{-1}AQ$ is a diagonal mattrix, where $Q = [y_1, y_2, \dots, y_n]$.

Exercise P259, Ex13 **How to find the eigenvectors of a $T \in L(V)$.**

Suppose $\dim V = n$ and β is an ordered basis for V. For a vector $v \in V$,

v is an eigenvector of T corresponding to λ

$\Leftrightarrow$ $[v]_\beta$ is an eigenvector of $[T]_\beta$ corresponding to λ

pf: $Tv = \lambda v \Leftrightarrow [Tv]_\beta = [\lambda v]_\beta \underset{\text{Thm2.14 P91}}{\Leftrightarrow} [T]_\beta [v]_\beta = \lambda [v]_\beta$

Ex7 let $T \in L(P_2(\mathbb{R}))$ s.t. $T(f(x)) = f(x) + (x+1)f'(x)$.

(1) Find all eigenvectors of T.

(2) Is T diagonalizable?

Sol: Let $\beta = \{1, x, x^2\}$

(1) $[T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ $\quad ev([T]_\beta) = \{1, 2, 3\}$

$$([T]_\beta - I)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{matrix} x_2 = 0 \\ x_2 + 2x_3 = 0 \\ 2x_3 = 0 \end{matrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}$$

for some $t \neq 0$

So $E_1 = \{ t : t \in \mathbb{R} \setminus \{0\} \}$

$$([T]_\beta - 2I)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} \text{ for some } t \neq 0$$

$E_2 = \{ t + tx : t \in \mathbb{R} \setminus \{0\} \}$

$$([T]_\beta - 3I)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ 2t \\ t \end{pmatrix} \text{ for some } t \neq 0$$

$E_3 = \{ t + 2tx + tx^2 : t \in \mathbb{R} \setminus \{0\} \}$

(2) Let $\gamma = \{1, 1+x, 1+2x+x^2\}$.

γ is an ordered basis for $P_2(\mathbb{R})$. Thus T is diagonalizable

and $[T]_\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

END.

eigenspace of A (or T)

Def: Let $A \in M_{n\times n}(F)$ and $\lambda \in ev(A)$.
The set $E_\lambda^A = \{ x \in M_{n\times 1}(F) : Ax = \lambda x \}$ is called the eigenspace of A corresponding to the eigenvalue λ.

Def: Let $T \in L(V)$ and $\lambda \in ev(T)$.
The set $E_\lambda^T = \{ x \in V : Tx = \lambda x \}$ is called the eigenspace of T correspondi[n]g the eigenvalue λ.

Fact: Let $\beta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an ordered basis for V.
Then

$$E_\lambda^T = \left\{ \sum_{i=1}^n x_i \alpha_i : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in E_\lambda^{[T]_\beta} \right\}$$

Geometric multiplicity
Algebraic multiplicity

Def: let $A \in M_{n\times n}(F)$ ($T \in L(V)$, V is a finite-dimensional vs over F).

- $\dim E_\lambda^A$ ($\dim E_\lambda^T$) is called the geometric multiplicity of λ
- The (algebraic) multiplicity of λ is the largest positive integer k for which $(x-\lambda)^k$ is a factor of $P_A(x)$ ($P_T(x)$).

Ex4 p265 $T \in L(\mathbb{R}^3)$ s.t. $T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 4a_1 + a_3 \\ 2a_1+3a_2+2a_3 \\ a_1 + 4a_3 \end{pmatrix}$

① Determine the eigenspace of T corresponding to each eigenvalue.

② Find the algebraic and geometric multiplicity for each eigenvalue.

Sol: $\boxed{\text{Step1}}$ Find ev(T):

let β = the standard ordered basis for $\mathbb{R}^3$.

$$[T]_\beta = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}. \quad \text{let } A = [T]_\beta.$$

$$P_A(x) = \det(A - xI) = -(x-5)(x-3)^2$$

so algebraic multiplicity $(5) = 1$

am$(3) = 2$

$\boxed{\text{Step2}}$

$$E_5^T = \{ x \in \mathbb{R}^3 : Tx = 5x \}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \left(\begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 5 & & \\ & 5 & \\ & & 5 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \Rightarrow \dim E_5^T = 1$$

$\boxed{\text{step3}}$ $E_3^T = \left\{ r \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : r, s \in \mathbb{R} \right\}.$

$\Rightarrow \dim E_3^T = 2.$

Thm 5.7 p264 Suppose $A \in M_{n\times n}(F)$ and $\lambda \in ev(A)$.

Then

$$1 \le \underbrace{(\text{geometric multiplicity of } \lambda)}_{}^{gm(\lambda)}$$

$$\le \underbrace{(\text{algebraic multiplicity of } \lambda)}_{am(\lambda)}$$

pf: Suppose $k = gm(\lambda)$.

$$\dim(E_\lambda^A) = k$$

$\Rightarrow \exists\ x_1, x_2, \dots x_k \in E_\lambda^A$ are l.i.

Let $Q = [x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{n-k}]$

be a nonsingular $n\times n$ matrix.

Then $AQ = [\lambda x_1, \lambda x_2, \dots \lambda x_k, z_1, z_2, \dots, z_{n-k}]$

$$= [x_1, x_2 \dots x_k, y_1, y_2 \dots y_{n-k}] \begin{array}{cc|l} \overbrace{\quad}^{k} & \overbrace{\quad}^{n-k} & \\ \multicolumn{2}{c}{\left[\begin{array}{c|c} \lambda \ddots \lambda & * \\ \hline O & * \end{array}\right]} & \begin{array}{l} \}k \\ \}n-k \end{array} \end{array}$$

z_i is a l.c. of $x_1, x_2 \dots x_k, y_1, \dots y_{n-k}$.

Thus $Q^{-1}AQ = \begin{bmatrix} \lambda I_k & M \\ O & N \end{bmatrix}$

p229 Ex21

$$P_A(x) = P_{Q^{-1}AQ}(x) = \det\begin{bmatrix} \lambda I_k - xI_k & M \\ O & N - xI_{n-k} \end{bmatrix}$$

$$= \det(\lambda I_k - xI_k)\det(N - xI_{n-k}) = (\lambda - x)^k \det(N - xI_{n-k})$$

QED

Thm 5.5 $T \in L(V)$ and $\lambda_1, \lambda_2, \ldots \lambda_k$ are distinct eigenvalues of T. Suppose $v_i \in E^T_{\lambda_i}$. $i=1,2,\cdots,k$.

Then $\{v_1, v_2, \ldots, v_k\}$ is l.i.

pf (sketch) By induction on k.

Suppose $a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0$.

We have

$$(T-\lambda_k I)(a_1v_1 + a_2v_2 + \cdots + a_kv_k) = 0$$

$$\Rightarrow a_1(\lambda_1-\lambda_k)v_1 + a_2(\lambda_2-\lambda_k)v_2 + \cdots + \underbrace{a_k(\lambda_k-\lambda_k)v_k}_{=0}$$

$$\Rightarrow \sum_{i=1}^{k-1} a_i(\lambda_i-\lambda_k)v_i = 0$$

$$\Rightarrow a_1 = a_2 = \cdots = a_{k-1} = 0$$ by induction hypothesis!

Thm 5.1 p246 $T \in L(V)$ and $\dim V = n$.

T is diagonalizable $\Longleftrightarrow$ $\exists$ an ordered basis β for V consisting of eigenvectors of T.

Pf: "$\Rightarrow$" T is diagonalizable

$\Rightarrow$ $\exists$ an ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$ for V

s.t. $[T]_\beta = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$

$\Rightarrow$ $Tv_i = \lambda_i v_i$ $i = 1, 2, \ldots, n$ and $v_i \neq 0$ $i = 1 \ldots n$

$\Rightarrow$ $v_1, v_2, \ldots, v_n$ are eigenvectors of T

"$\Leftarrow$" $\exists$ an ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$ for V s.t. $Tv_i = \lambda_i v_i$, $v_i \neq 0$ $i = 1, 2, \ldots n$.

$\Rightarrow$ $[T]_\beta = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$

$\Rightarrow$ T is diagonalizable.

QED

Thm 5.9 P268 Let V be a vs over F with $\dim V = n$.

Let $T \in L(V)$ with

$$P_T(x) = (-1)^n (x-\lambda_1)^{t_1}(x-\lambda_2)^{t_2} \cdots (x-\lambda_k)^{t_k}$$

s.t. $\lambda_1, \lambda_2, \ldots \lambda_k$ are distinct eigenvalues of T.

(a) T is diagonalizable $\Leftrightarrow$ $gm(\lambda_i) = am(\lambda_i)$ $\forall i$

(b) If T is diagonalizable and β_i is an ordered basis for $E^T_{\lambda_i}$ then $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered basis for V.

Pf: (a) "$\Rightarrow$"

T is diagonalizable

$\Rightarrow \exists$ an ordered basis β for V consisting of eigenvector of T

$$\Rightarrow n = \sum_{i=1}^{k} |\beta \cap E^T_{\lambda_i}| \le \sum_{i=1}^{k} \dim E^T_{\lambda_i}$$

$$= \sum_{i=1}^{k} gm(\lambda_i)$$

$$\le \sum_{i=1}^{k} am(\lambda_i) = n$$

Thus it must be $gm(\lambda_i) = am(\lambda_i)$ $\forall i$.

pf (Thm 5.9)

(a) "$\Leftarrow$"

Let β_i be an ordered basis for $E_{\lambda_i}^T$

$i = 1, 2, \cdots, k$.

Since
$$\sum_{i=1}^{k} |\beta_i| = \sum_{i=1}^{k} \dim E_{\lambda_i}^T = \sum_{i=1}^{k} gm(\lambda_i) = \sum_{i=1}^{k} am(\lambda_i) = n$$

and $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is l.i., we see

that β is an ordered basis for V

consisting of eigenvector of T.

QED.

How to show T is NOT diagonalizable (I)

Def p262 Let $p(x) \in P(F)$. We say that $p(x)$ splits over F if $\exists\ c, a_1, \dots, a_n \in F$ s.t.

$$p(x) = c(x-a_1)(x-a_2)\cdots(x-a_n).$$

Ex: $(x^2+1)(x-2)$ splits over $\mathbb{C}$

$(x^2+1)(x-2)$ does not split over $\mathbb{R}$

Thm p263 5.6 $T \in L(V)$, T is diagonalizable

$\Rightarrow$ $P_T(x)$ splits

Remark The converse of Thm 5.6 is false. See next Ex.

Ex: p264 Let $T \in L(P_2(\mathbb{R}))$ s.t. $T(f(x)) = f'(x)$.

(1) Does $P_T(x)$ split?

(2) Is T diagonalizable?

Sol: Let β be the standard ordered basis for $P_2(\mathbb{R})$.

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}. \quad P_{[T]_\beta}(x) = -x^3$$

eigenspace

$E_0^T = \{ f(x) : T(f(x)) = 0 \} = \{ a : a \in \mathbb{R} \}$

$\dim E_0^T = 1 \neq 3$. $\nexists$ basis for $P_2(\mathbb{R})$ consisting of eigenvectors of T.

Test for Diagonalization

- $T \in L(V)$, $\dim V = n$. V is a vs over F.

Fact T is diagonalizable.

$\Leftrightarrow$ ① $P_T(x)$ splits over F

② $\lambda \in ev(T) \Rightarrow gm(\lambda) = am(\lambda)$.

$\Leftrightarrow$ ① $P_T(x)$ splits over F

② $\lambda \in ev(T) \Rightarrow n - \text{rank}(T - \lambda I) = am(\lambda)$.

Ex Let $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \in M_{3\times3}(\mathbb{R})$.

Is A diagonalizable?

Sol: $P_A(x) = -(x-4)(x-3)^2$ which splits.

$$\text{nullity}(A-4I) = \text{nullity}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 3 - \text{rank}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1$$

$$\text{nullity}(A-3I) = 3 - \text{rank}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1$$

$am(4) = 1 = 1 = gm(1)$

$am(3) = 2 \neq 1 = gm(3)$. So A is not diagonalizable!

Ex6 自己看！

Ex7 Let $A=\begin{pmatrix}0&-2\\1&3\end{pmatrix}$. Find A^{2006}.

sol:

$p_A(x)=(x-1)(x-2)$.

$E_1^A=\left\{t\begin{pmatrix}-2\\1\end{pmatrix}:t\in\mathbb{R}\right\}$

$E_2^A=\left\{t\begin{pmatrix}-1\\1\end{pmatrix}:t\in\mathbb{R}\right\}$

Let $Q=\begin{pmatrix}-2&-1\\1&1\end{pmatrix}$ then $Q^{-1}=\begin{pmatrix}-1&-1\\1&2\end{pmatrix}$.

$A=Q\begin{pmatrix}1&0\\0&2\end{pmatrix}Q^{-1}$

$$A^{2006}=Q\begin{pmatrix}1&0\\0&2^{2006}\end{pmatrix}Q^{-1}$$

$$=\begin{pmatrix}2-2^{2006}&2-2^{2007}\\-1+2^{2006}&-1+2^{2007}\end{pmatrix}.$$

END.