Direct Sum

Def p276 Let $W_1, \cdots, W_k$ be subspaces of V.

- $W_1 + \cdots + W_k \overset{def}{=} \{ u_1 + u_2 + \cdots + u_k : u_i \in W_i,\ 1 \le i \le k \}$

  $\sum_{i=1}^{k} W_i$

- We call V is a direct sum of $W_1, W_2, \cdots, W_k$, and write $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ if

  ① $V = W_1 + W_2 + \cdots + W_k$, and

  ② $W_j \cap \sum_{i \ne j} W_i = \{0\}$ for $1 \le j \le k$.

  zero vector

Thm 5.10 p276 Let $W_1, \cdots, W_k$ be subspaces of V and $\dim V < \infty$. The followings are equivalent.

(a) $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$

(b) $V = W_1 + W_2 + \cdots + W_k$ and for any $u_1 \in W_1, u_2 \in W_2, \cdots, u_k \in W_k$ if $u_1 + u_2 + \cdots + u_k = 0$ then $u_i = 0$ for all i.

(c) $\forall\, v \in V$ can be uniquely written as $v = u_1 + u_2 + \cdots + u_k$ where $u_i \in W$.

(d) If γ_i is an ordered basis for W_i, $\forall i$, ~~then~~ then $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is an ordered basis for V.

pf (a) $\Rightarrow$ (b) Obviously see page 7, 8

(b) $\Rightarrow$ (c) Obviously!

(c) $\Rightarrow$ (d) Obviously!

(d) $\Rightarrow$ (a)

$\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_K$ is an ordered basis for V

$\Rightarrow V = W_1 + W_2 + \cdots + W_K$

Now assume (w.l.o.g) $\exists\, v_1 \in W_1 \setminus \{0\}$ s.t.

$v_1 = v_2 + v_3 + \cdots + v$ where

$v_2 \in W_{j_2} \setminus \{0\}$, $v_3 \in W_{j_3} \setminus \{0\}$, $\ldots$, $v_\ell \in W_{j_\ell} \setminus \{0\}$

$2 \leq j_2 < j_3 < \cdots < j_\ell \leq K$.

We can find ordered basis γ_2 for W_{j_2}, $\cdots$

γ_ℓ for W_{j_ℓ} s.t.

$v_2 \in \gamma_2$, $v_3 \in \gamma_3$, $\ldots$, $v_\ell \in \gamma_\ell$.

Note that v can be expressed in two ways.

That's a contradiction. see p 43.

QED

Thm 5.11 let $T \in L(V)$ and $\dim V < \infty$ (over F). Then

T is diagonalizable $\Longleftrightarrow$ V is the direct sum of the eigenspaces of T

pf: T is diagonalizable

$\Longleftrightarrow$ ① $p_T(x)$ splits over F

② $\lambda \in ev(T) \Rightarrow gm(\lambda) = am(\lambda)$

$\overset{\text{Thm 5.9}}{\Longleftrightarrow}$ $p_T(x) = (-1)^n (x-\lambda_1)^{t_1}(x-\lambda_2)^{t_2} \cdots (x-\lambda_k)^{t_k}$

where $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct eigenvalues of T, and $\dim E^T_{\lambda_i} = t_i$, $i = 1, 2, \cdots, k$.

$\overset{\text{Thm 5.9}}{\Longleftrightarrow}$ If β_i is an ordered basis for $E^T_{\lambda_i}$ then $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered basis for V

$\overset{\text{Thm 5.10}}{\Longleftrightarrow}$ $V = E^T_{\lambda_1} \oplus E^T_{\lambda_2} \oplus \cdots \oplus E^T_{\lambda_k}$

Matrix Limits

Def: $B, A_1, A_2, \cdots \in M_{n\times p}(\mathbb{C})$

If $\lim_{m\to\infty} (A_m)_{ij} = B_{ij}$ for all $1\le i\le n,\ 1\le j\le p$

then we write $\lim_{m\to\infty} A_m = B.$

$A_1, A_2, \cdots$ converge to B

Ex 1: $\lim_{m\to\infty} \begin{pmatrix} 1-\frac{1}{m} & \left(-\frac{3}{4}\right)^m & \frac{3m^2}{m^2+1} + i\left(\frac{2m+1}{m-1}\right) \\ \left(\frac{i}{2}\right)^m & 2 & \left(1+\frac{1}{m}\right)^m \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 3+2i \\ 0 & 2 & e \end{pmatrix}$

Thm 5.12: $A_1, A_2, \ldots \in M_{n\times p}(\mathbb{C})$ s.t.

$\lim_{m\to\infty} A_m = B$. Then for any $P \in M_{r\times n}(\mathbb{C})$,

$Q \in M_{p\times s}(\mathbb{C})$,

$\lim_{m\to\infty} PA_m = PB$, $\lim_{m\to\infty} A_m Q = BQ$.

Thm 5.14 Let $A \in M_{n\times n}(\mathbb{C})$ s.t.

① $\text{ev}(A) \subseteq \{\lambda \in \mathbb{C} : |\lambda|<1 \text{ or } \lambda=1\}$

② A is diagonalizable.

Then $\lim_{m\to\infty} A^m$ exists.

pf:

$$\lim_{m\to\infty} A^m = \lim_{m\to\infty}\left(Q\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_n \end{bmatrix} Q^{-1}\right)^m$$

$$= \lim_{m\to\infty} Q \begin{bmatrix} \lambda_1^m & & 0 \\ & \lambda_2^m & \\ 0 & & \ddots & \\ & & & \lambda_n^m \end{bmatrix} Q^{-1}$$

已知☆這 matrix limit 存在才有這等式

$$= Q\underbrace{\left(\lim_{m\to\infty} \begin{bmatrix} \lambda_1^m & & 0 \\ & \lambda_2^m & \\ 0 & & \ddots & \\ & & & \lambda_n^m \end{bmatrix}\right)}_{\star} Q^{-1}$$

$$= Q \begin{bmatrix} b_1 & & 0 \\ & b_2 & \\ 0 & & \ddots & \\ & & & b_n \end{bmatrix} Q^{-1}$$

where $b_i = \begin{cases} 0 & \text{if } |\lambda_i|<1 \\ 1 & \text{if } \lambda_i = 1 \end{cases}$

QED

Ex: Let $A = \begin{pmatrix} \frac{7}{4} & -\frac{9}{4} & -\frac{15}{4} \\ \frac{3}{4} & \frac{7}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{9}{4} & -\frac{11}{4} \end{pmatrix}$.

Note that

$$\underbrace{\begin{pmatrix} 1 & 3 & -1 \\ -3 & -2 & 1 \\ 2 & 3 & -1 \end{pmatrix}}_{Q} \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & \frac{1}{4} \end{pmatrix} Q^{-1} = A$$

$$\lim_{m\to\infty} A^m = Q \lim_{m\to\infty} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}^m Q^{-1}$$

$$= Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^{-1}$$

$$= \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ -2 & 0 & 2 \end{pmatrix}$$

e^A

Fact P312, EX22 If $A \in M_{n\times n}(\mathbb{C})$ is diagonalizable. Then the limit $\lim_{m\to\infty} B_m$ exists, where $B_m = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^m}{m!}$. We write $e^A \overset{\text{def}}{=} \lim_{m\to\infty} B_m$.

pf:

$$A = Q^{-1} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{L} Q$$

$$B_m = Q^{-1}\left(I + L + \frac{L^2}{2!} + \frac{L^3}{3!} + \cdots + \frac{L^m}{m!} \right) Q$$

$$= Q^{-1} \begin{pmatrix} 1+\lambda_1+\frac{\lambda_1^2}{2!}+\frac{\lambda_1^3}{3!}+\cdots+\frac{\lambda_1^m}{m!} & & & \\ & 1+\lambda_2+\frac{\lambda_2^2}{2!}+\frac{\lambda_2^3}{3!}+\cdots+\frac{\lambda_2^m}{m!} & & \\ & & \ddots & \\ & & & 1+\lambda_n+\frac{\lambda_n^2}{2!}+\frac{\lambda_n^3}{3!}+\cdots+\frac{\lambda_n^m}{m!} \end{pmatrix} Q$$

$$\lim_{m\to\infty} B_m = Q^{-1} \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix} Q$$

QED

Def: $A \in M_{n\times n}(\mathbb{C})$. $A = [a_{ij}]$

$$rs_i(A) \overset{\text{def}}{=\!=} \sum_{j=1}^{n} |a_{ij}|$$

$$cs_j(A) \overset{\text{def}}{=\!=} \sum_{i=1}^{n} |a_{ij}|$$

row sum of A $\quad rs(A) \overset{\text{def}}{=\!=} \max\{ rs_i(A) : 1 \le i \le n \}$

column sum of A $\quad cs(A) \overset{\text{def}}{=\!=} \max\{ cs_j(A) : 1 \le j \le n \}$

Def: Gerschgorin disk

$$C_i \overset{\text{def}}{=\!=} \{ z \in \mathbb{C} : |z - a_{ii}| < rs_i(A) - |a_{ii}| \}$$

__Ex:__ $A = \begin{pmatrix} 1+2i & 1 \\ 2i & -3 \end{pmatrix}$

$$C_1 = \{ z \in \mathbb{C} : |z - (1+2i)| < \overbrace{rs_1(A)}^{|(1+2i)|+|1|} - |(1+2i)| \}$$

$$C_2 = \{ z \in \mathbb{C} : |z - (-3)| < rs_2(A) - |(-3)| \}$$

__Note__: Sometimes we use the following notation

$|||A|||_1 = cs(A)$, $\quad |||A|||_\infty = rs(A)$

Gerschgorin's Disk Thm

Thm 5.16 P296 $A \in M_{n\times n}(\mathbb{C})$.

Then every eigenvalue of A is contained in a Gerschgorin disk. i.e.

$$\lambda \in ev(A) \Rightarrow |\lambda - a_{kk}| \le \underbrace{rs_k(A) - |a_{kk}|}_{\sum_{j\neq k} |a_{kj}|}$$

for some k

pf: $\lambda \in ev(A) \Rightarrow$ eigenvector $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ s.t.

$$Av = \lambda v$$

Let v_k be one of coordinates of v s.t.

$$|v_k| = \max\{|v_1|, |v_2|, \cdots, |v_n|\}$$

$$\begin{aligned} |\lambda - a_{kk}||v_k| &= |\lambda v_k - a_{kk} v_k| \\ &= \Big|\sum_{j\neq k} a_{kj} v_j\Big| \\ &\le \sum_{j\neq k} |a_{kj}||v_k| \end{aligned}$$

So $|\lambda - a_{kk}| \le \sum_{j\neq k} |a_{kj}|$ (note: $|v_k| \neq 0$).

QED

Corollary p297 $A \in M_{n\times n}(\mathbb{C})$.

$$\lambda \in ev(A) \Rightarrow |\lambda| \le \min\{rs(A),\ cs(A)\}$$

pf: $|\lambda| \le |\lambda - a_{kk}| + |a_{kk}|$ for some k.

$$\le \Big(\sum_{j \ne k} |a_{kj}|\Big) + |a_{kk}| \le rs(A)$$

$\lambda \in ev(A^t) \Rightarrow |\lambda| \le cs(A)$.

QED

Thm 5.18 $A = [a_{ij}] \in M_{n\times n}(\mathbb{C})$ s.t. $a_{ij} > 0$.

$\lambda \in ev(A)$ s.t. $|\lambda| = rs(A) \Rightarrow \lambda = rs(A)$ and

$\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$ is a basis for E_λ^A

pf: Suppose $Av = \lambda v$, $v = \begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix} \ne \underset{\sim}{0}$.

Say, $|v_k| = \max\{|v_1|, \cdots, |v_n|\}$.

$$rs(A)\,|v_k| = |\lambda v_k| = \Big|\sum_{j=1}^{n} a_{kj} v_j\Big| \le \sum_{j=1}^{n} |a_{kj}|\,|v_j|$$

$$\le \sum_{j=1}^{n} |a_{kj}|\,|v_k| \le rs(A)\,|v_k|$$

Thus. $|v_j| = |v_k|$, $j = 1, 2, \cdots, n$ ($\because |a_{ij}| > 0\ \forall i,j$),

and $a_{kj} v_j = c_j z$, $j = 1, 2, \cdots, n$, where $c_j > 0$, $j = 1, 2, \cdots, n$. (w.l.o.g. assume $|z| = 1$)

pf (Thm 5.18 continued)

Thus $|u_k| = \frac{c_j}{a_{kj}}$ for $j = 1, 2, \cdots, n$ ($\because c_j > 0$, $a_{kj} > 0$)

Therefore

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = |u_k| \begin{pmatrix} z \\ z \\ \vdots \\ z \end{pmatrix} = |u_k| z \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$ and hence

$$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in E_\lambda^A.$$

$$A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \Rightarrow \lambda > 0 \quad (\because a_{ij} > 0)$$

$$\Rightarrow \lambda = rs(A).$$

QED

Invariant Subspace

Def: let $T \in L(V)$. A subspace W of V is called T-invariant subspace of V if $T(W) \subseteq W$.

i.e. $T(\omega) \in W$ for any $\omega \in W$

Let $x \in V \setminus \{0\}$. The following subspace of V

$$W = \mathrm{span}(\{x, T(x), T^2(x), \dots\})$$

is called the T-cyclic subspace of V generated by x.

Note: W is T-invariant subspace of V.

Notation: T_W is the restriction of T to W. see p552

Thm 5.21: $T \in L(V)$, $\dim V < \infty$, W is a T-invariant subspace of V.

Then $P_{T_W}(x) \mid P_T(x)$.

pf: Let W: $\gamma = \{v_1, v_2, \dots, v_k\}$ ordered basis for W | $v_{k+1}, \dots, v_n$

$\beta = \{v_1, \dots, v_n\}$ is an ordered basis for V

pf (Continued)

$$P_T(x) = P_{[T]_\beta}(x)$$

$$= \det\left(\begin{bmatrix} [T_W]_\gamma & B_2 \\ 0 & B_3 \end{bmatrix} - x I_n \right)$$

$$= \det\left([T_W]_\gamma - x I_k\right) \det\left(B_3 - x I_{n-k}\right)$$

$$= P_{T_W}(x) \det\left(B_3 - x I_{n-k}\right)$$

QED

Thm 5.22 Let $T \in L(V)$ and $\dim V < \infty$.

Let $W = \text{Span}(\{v, T(v), T^2(v), \cdots\})$, $v \neq 0$.

let $k = \dim W$. Then

T-cyclic subspace of V generated by v

(a) $\{v, T(v), T^2(v), T^3(v), \cdots, T^{k-1}(v)\}$ is a basis for W.

(b) If $\displaystyle\sum_{i=0}^{k-1} a_i T^i(v) + T^k(v) = 0$ (where $T^0 = I$)

then $P_{T_W}(x) = (-1)^k \left(\displaystyle\sum_{i=0}^{k-1} a_i x^i + x^k \right)$

I3

pf: (a) $j \overset{\text{def}}{=\!=} \max\left\{ i : \{v, T(v), T^2(v), \cdots, T^{i-1}(v)\} \text{ is l.i.} \right\}$

($j < \infty$ since $\dim V < \infty$)

$$Z \overset{\text{def}}{=\!=} \operatorname{span}\Big(\underbrace{\{v, T(v), T^2(v), \cdots, T^{j-1}(v)\}}_{\beta} \Big)$$

Claim. $W = Z$

pf: $\forall k$, $T^k(v) \in Z$

$\Rightarrow \operatorname{span}(\{v, T(v), T^2(v), \dots\}) \subseteq Z$ QED

$j = \dim Z = \dim W \overset{\text{hypothesis}}{=} k$. So (a) holds.

(b) let β be an ordered basis for W.

$$[T_W]_\beta = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & & 0 & -a_1 \\ 0 & 1 & 0 & & 0 & -a_2 \\ 0 & 0 & 1 & & 0 & \vdots \\ 0 & 0 & 0 & \ddots & 0 & \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & & 1 & -a_{k-1} \end{bmatrix}_{k \times k}$$

$$P_{[T_W]_\beta} = \det \begin{bmatrix} -x & & & & & -a_0 \\ 1 & -x & & & & -a_1 \\ & 1 & -x & & & -a_2 \\ & & 1 & \ddots & & \vdots \\ & & & & -x & -a_{k-2} \\ & & & & 1 & -a_{k-1}-x \end{bmatrix}$$

$$= (-1)^k (a_0 + a_1 x + a_2 x^2 + \cdots + a_{k-1} x^{k-1} + x^k)$$

QED

Compute P_{T_W} in two ways

Ex $T \in L(\mathbb{R}^3)$ s.t. $T(a,b,c) = (-b+c,\ a+c,\ 3c)$

let $v_1 = (1,0,0)$.

① Compute T-cyclic subspace W of V generated by v

② Compute $P_{T_W}(x)$.

Sol: ① $T(v_1) = \underbrace{(0,1,0)}_{v_2}$. $T^2(v_1) = T(v_2) = -v_1$

$T^3(v_1) = T(-v_1) = -v_2$, $T^4(v_1) = T(-v_2) = v_1$.

so $W = \text{span}(\{v_1, v_2\})$.

(2-1) Note that $\dim W = 2$ and $T^0(v_1) + 0\,T(v_1) + T^2(v_1) = 0$.

$$\text{Thm 5.22} \Rightarrow P_{T_W}(x) = (-1)^2\left(1x^0 + 0x + x^2\right) = x^2+1$$

(2-2) Let $\beta = \{v_1, v_2\}$ be an ordered basis for W.

$$P_{T_W}(x) = P_{[T_W]_\beta}(x) = \det\begin{bmatrix} -x & -1 \\ 1 & -x \end{bmatrix} = x^2+1.$$

END

Cayley-Hamilton Thm

Thm 5.23 let $T \in L(V)$ and $\dim V < \infty$.

Then $P_T(T)$ is the zero transformation from V to itself. i.e. T satisfies its characteristic equation.

Pf: For $v \in V \setminus \{0\}$.

let $W = \text{span}(\{v, T(v), T^2(v), \dots\})$ and $\dim W = k$.

$*$ Thm 5.22(a) $\Rightarrow \exists\, a_0, a_1, \dots, a_{k-1}$ s.t.

$$a_0 v + a_1 T(v) + \cdots + a_{k-1} T^{k-1}(v) + T^k(v) = 0$$

zero vector

$\star$ Thm 5.22(b) $\Rightarrow P_{T_W}(x) = (-1)^k (a_0 + a_1 x + a_2 x^2 + \cdots + a_{k-1} x^{k-1} + x^k)$

Thm 5.21 $\Longrightarrow P_{T_W}(x) \mid P_T(x)$ ($\because$ W is T-invariant)

$$\Longrightarrow P_T(x) = q(x) P_{T_W}(x)$$

$$\Longrightarrow P_T(T)(v) = q(T) P_{T_W}(T)(v)$$

$$= q(T)\left(P_{T_W}(T)(v)\right)$$

$$= q(T)\left((-1)^k (a_0 v + a_1 T(v) + a_2 T^2(v) + \cdots + a_{k-1} T^{k-1}(v) + T^k(v))\right)$$

$$\overset{\text{by } *}{=} q(T)(0)$$

$$= 0$$

QED

Ex7 P318 let $T \in L(\mathbb{R}^2)$ s.t. $T(a,b) = (a+2b, -2a+b)$

let $\beta = \{\underset{\binom{1}{0}}{e_1}, \underset{\binom{0}{1}}{e_2}\}$. Find $T^5 + T^3 - 4T + 4I$

Sol: $P_T(x) = P_{[T]_\beta}(x) = \det([T]_\beta - xI) = \det\begin{pmatrix} 1-x & 2 \\ -2 & 1-x \end{pmatrix}$

$= x^2 - 2x + 5$

Note that

$x^5 + x^3 - 4x + 4 = (x^3 + 2x^2 - 10)(x^2 - 2x + 5) + (54 - 24x)$

So $(T^5 + T^3 - 4T + 4I)(a,b)$

$= (T^3 + 2T^2 - 10I)\big((T^2 - 2T + 5I)(a,b)\big) + (54I - 24T)(a,b)$

$= (T^3 + 2T^2 - 10I)(0,0) + 54(a,b) - 24(a+2b, -2a+b)$

$= (30a - 48b, 48a + 30b)$

Corollary: $A \in M_{n\times n} \Longrightarrow P_A(A) =$ $n\times n$ zero matrix.