Inner Products

note: 本章中 $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Def p329 An **Inner product space** V is a vector space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) endowed with a specific function $\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F}$ s.t. (called inner product on V)

for $\forall\, x, y, z \in V$ and $c \in \mathbb{F}$ the following hold:

ⓐ $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

ⓑ $\langle cx, y \rangle = c \langle x, y \rangle$

ⓒ $\overline{\langle x, y \rangle} = \langle y, x \rangle$ ← complex conjugation

ⓓ $\langle x, x \rangle > 0$ if $x \neq 0$

Def p331 Let $A \in M_{m \times n}(\mathbb{F})$.

conjugate transpose of $A \overset{def}{=\!=} A^*$
(or adjoint)

where $(A^*)_{ij} = \overline{A_{ji}}$ $\forall\, i, j$.

Ex: $A = \begin{pmatrix} i & 1+2i \\ 2 & 3+4i \end{pmatrix} \Longrightarrow A^* = \begin{pmatrix} -i & 2 \\ 1-2i & 3-4i \end{pmatrix}$

Let V be an inner product space over F.

Def: For $v \in V$, $\|v\| \overset{def}{=} \sqrt{\langle v, v \rangle}$

the norm of x

(the length of x)

Thm 6.1 + 6.2

For $x, y, z \in V$ and $c \in F$

ⓐ $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

ⓑ $\langle x, cy \rangle = \bar{c} \langle x, y \rangle$

ⓒ $\langle x, 0 \rangle = \langle 0, x \rangle = 0$

ⓓ $\langle x, x \rangle = 0 \iff x = 0$

ⓔ If $\langle x, y \rangle = \langle x, z \rangle$ for all $x \in V$ then $y = z$.

ⓕ $\|cx\| = |c| \cdot \|x\|$

$z = a+bi \Rightarrow |z| \overset{def}{=} \sqrt{a^2+b^2}$

note: $z\bar{z} = |z|^2$

ⓖ $\|x\| = 0 \iff x = 0$,

ⓗ $\|x\| \geq 0$

ⓘ $|\langle x, y \rangle| \leq \|x\| \, \|y\|$ (Cauchy-Schwarz Ineq.)

ⓙ $\|x+y\| \leq \|x\| + \|y\|$ (Triangle Inequality)

Pf (i) let $y \in V \setminus \{0\}$.

$$0 \le \langle x - cy, x - cy \rangle$$

$$= \langle x, x \rangle - \bar{c}\langle x, y \rangle - c\langle y, x \rangle + |c|^2 \langle y, y \rangle$$

$$= \|x\|^2 - \bar{c}\langle x, y \rangle - \overline{\bar{c}\langle x, y \rangle} + |c|^2 \|y\|^2$$

Set $c = \frac{\langle x, y \rangle}{\|y\|^2}$. Note that $\bar{c}\langle x, y \rangle = \frac{|\langle x, y \rangle|^2}{\|y\|^2} \in \mathbb{R}$

Then $\|x\|^2 - 2\frac{|\langle x, y \rangle|^2}{\|y\|^2} + \frac{|\langle x, y \rangle|^2}{\|y\|^2} \ge 0$

Therefore $\|x\|^2 \|y\|^2 \ge |\langle x, y \rangle|^2$

(ii) $\|x + y\|^2 = \langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle$

$$= \|x\|^2 + 2\overbrace{\mathrm{Re}\langle x, y \rangle}^{\text{real part of } \langle x, y \rangle} + \|y\|^2$$

$$\le \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2$$

Cauchy-Schwarz Ineq.

$$\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

Ex: P330 For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{F}^n$, define $\langle x, y \rangle = \sum_{i=1}^n x_i \overline{y_i}$. Then $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{F}^n$.

Def: let V be an inner product space, $x, y \in V$.

- If $\langle x, y \rangle = 0$ then we say that x and y are orthogonal.
- $S \subseteq V$ is called orthogonal if $\forall$ two distinct vectors in S are orthogonal.
- unit vector u : i.e. $\|u\| = 1$
- $S \subseteq V$ is orthonormal if
  - ⓐ $\|u\| = 1$ for $\forall u \in S$.
  - ⓑ S is orthogonal.

Ex: Let $H = \{ f : f$ is a continuous complex-valued fun. defined on $[0, 2\pi] \}$.
be an inner product space with

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{g(t)} \, dt.$$

Let $S = \{ f_n : n \in \mathbb{Z} \}$

where $f_n(t) : [0, 2\pi] \longrightarrow \mathbb{C}$ s.t. $f_n(t) = e^{int}$.

Show that S is orthonormal !!

Ex: Two inequalities

Show (a) $\left| \sum_{i=1}^{n} a_i \overline{b_i} \right| \leq \left(\sum_{i=1}^{n} |a_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |b_i|^2 \right)^{\frac{1}{2}}$

(b) $\left[\sum_{i=1}^{n} |a_i + b_i|^2 \right]^{1/2} \leq \left[\sum_{i=1}^{n} |a_i|^2 \right]^{1/2} + \left[\sum_{i=1}^{n} |b_i|^2 \right]^{1/2}$

pf: let $x = (a_1, \cdots, a_n)$, $y = (b_1, \cdots b_n)$ be two vector in the inner product space $\mathbb{C}^n$ with standard inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} a_i \overline{b_i}$$

(see p330)

$|\langle x, y \rangle| \leq \|x\| \, \|y\| \Longrightarrow$ (a) holds

$\|x + y\| \leq \|x\| + \|y\| \Longrightarrow$ (b) holds

QED

Def: An ordered basis β of an inner product space V (over $\mathbb{R}$ or $\mathbb{C}$) is an orthonormal basis if β is orthonormal.

Thm 6.3 Let V be an inner product space and $S = \{v_1, v_2, \cdots, v_k\} \subseteq V \setminus \{0\}$.

(a) If S is orthogonal and $y = \sum_{i=1}^{k} a_i v_i$ then $a_i = \dfrac{\langle y, v_i \rangle}{\| v_i \|^2}$ $\forall i$.

(b) If S is orthogonal then S is l.i.

Gram-Schmidt Orthogonalization Process

Thm Let $\{\beta_1, \beta_2, \cdots, \beta_m\}$ be a l.i. set in the inner product space $(V, F, \langle \cdot, \cdot \rangle)$. Construct a set of vectors

$$\alpha_1 = \beta_1$$

$$\alpha_2 = \beta_2 - \frac{\langle \beta_2, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle} \alpha_1$$

$$\vdots$$

$$\alpha_m = \beta_m - \sum_{k=1}^{m-1} \frac{\langle \beta_m, \alpha_k \rangle}{\langle \alpha_k, \alpha_k \rangle} \alpha_k$$

then (1) $\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ is orthogonal.

(2) $\operatorname{Span}\{\beta_1, \cdots, \beta_k\} = \operatorname{span}\{\alpha_1, \alpha_2 \ldots \alpha_k\}$.

$k = 1, 2, 3, \cdots, m$

pf (Gram-Schmidt Process)

(1) (Basis) $\langle \alpha_2, \alpha_1 \rangle = \langle \beta_2 - \frac{\langle \beta_2, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle} \alpha_1, \alpha_1 \rangle$

$= \langle \beta_2, \alpha_1 \rangle - \langle \beta_2, \alpha_1 \rangle = 0$

(Induction Part) Assume that $\{\alpha_1, \cdots \alpha_{t-1}\}$ is orthogonal. To show $\{\alpha_1, \cdots, \alpha_t\}$ is orthogonal.

For $1 \le j \le t-1$,

$\langle \alpha_t | \alpha_j \rangle$

$$= \langle \beta_t - \sum_{k=1}^{t-1} \frac{\langle \beta_t, \alpha_k \rangle}{\langle \alpha_k, \alpha_k \rangle} \alpha_k, \alpha_j \rangle$$

$$= \langle \beta_t, \alpha_j \rangle - \frac{\langle \beta_t, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \langle \alpha_j, \alpha_j \rangle = 0.$$

QED

Thm 6.5 Let V be an inner product space with $0<\dim V<\infty$.

① V has an orthonormal basis $\beta=\{v_1, v_2, \cdots, v_n\}$

② If $T\in L(V)$ then $[T]_\beta=(a_{ij})_{n\times n}$.

where $a_{ij}=\langle Tv_j, v_i\rangle$.

pf: ② Suppose $Tv_j=\sum_{k=1}^{n} a_{kj}v_k$.

$\langle Tv_j, v_i\rangle = a_{ij}\langle v_i, v_i\rangle = a_{ij}$

QED

Orthogonal Complement

Def: Let S is a nonempty subset of an inner product space $(V, F, \langle\cdot,\cdot\rangle)$.

"S perp"

$$S^{\perp} \stackrel{\text{def}}{=} \{x \in V : \langle x, y\rangle = 0 \text{ for all } y \in S\}.$$

orthogonal complement of S

Thm 6.6 Let W be a subspace of an inner product space V, where W has an orthonormal basis $\{u_1, u_2, \cdots, u_k\}$, then, for $y \in V$,

① $\exists!$ vectors $u \in W$ and $z \in W^{\perp}$ s.t. $y = u + z$

* u is called the orthogonal projection of y on W.

② $u = \sum_{i=1}^{k} \langle y, u_i\rangle u_i$

③ For any $x \in W$, $\|y - x\| \geq \|y - u\|$

"=" holds $\Leftrightarrow$ $x = u$

pf: ① Suppose $y = u + z = u' + z'$ and $u, u' \in W$, $z, z' \in W^{\perp}$. Then $u - u' = z' - z \in W^{\perp} \cap W$.

That is $u - u' = z' - z = 0$.

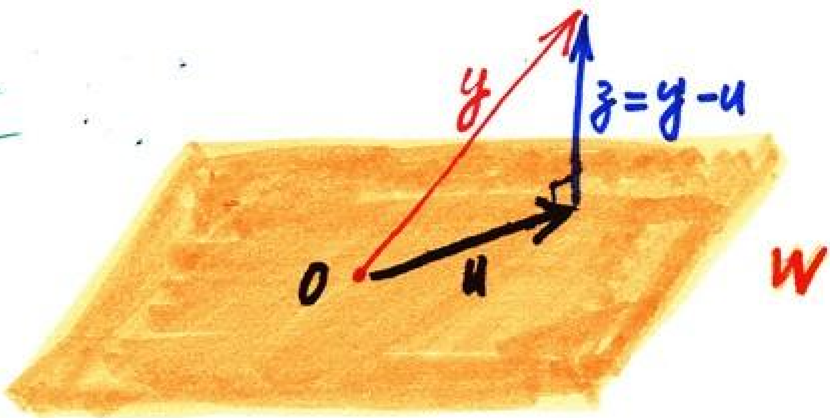
pf (continued)

② Let $z = y - \sum_{i=1}^{k} \langle y, u_i \rangle u_i$.

Then

$$\langle z, u_t \rangle = \langle y, u_t \rangle - \langle y, u_t \rangle \langle u_t, u_t \rangle$$
$$= 0$$

③



$$\|y-x\|^2 = \|u+z-x\|^2$$
$$= \|u-x+z\|^2$$
$$= \langle (u-x)+z, (u-x)+z \rangle$$
$$= \|u-x\|^2 + \|z\|^2 \geq \|z\|^2 = \|y-u\|^2$$

$$\|y-x\|^2 = \|y-u\|^2 \Leftrightarrow \|u-x\|^2 = 0$$
$$\Leftrightarrow u = x$$

QED

Note: 讀者可以先証 Thm6.7 再回頭 Thm 6.6.

Ex: Let $V = P(\mathbb{R})$ be an inner product space with $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$.

Let β be the standard ordered basis for the subspace $P_2(\mathbb{R})$ of V.

① Use the Gram-Schmidt process to replace β by an orthogonal basis $\{v_1, v_2, v_3\}$ for $P_2(\mathbb{R})$.

② Use $\{v_1, v_2, v_3\}$ to obtain an orthonormal basis $\{u_1, u_2, u_3\}$ for $P_2(\mathbb{R})$.

③ Represent $f(x) = 1+2x+3x^2$ as a l.c. of the vectors in $\{u_1, u_2, u_3\}$.

④ $f(x) = x^3$ is a vector in the inner product space $P_3(\mathbb{R})$ with inner product $\langle \cdot, \cdot \rangle$.

Compute the orthogonal projection $f_1(x)$ of $f(x)$ on $P_2(\mathbb{R})$.

Sol:

① $v_1 = 1$

$$v_2 = x - \frac{\langle x, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = x - \frac{0}{2} = x$$

$$v_3 = x^2 - \frac{\langle x^2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle x^2, v_2\rangle}{\langle v_2, v_2\rangle} v_2 = x^2 - \frac{1}{3}$$

② $$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{\int_{-1}^{1} 1^2\, dt}} = \frac{1}{\sqrt{2}}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{x}{\sqrt{\int_{-1}^{1} t^2\, dt}} = \sqrt{\frac{3}{2}}\, x$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{-1}^{1} (t^2 - \frac{1}{3})^2 dt}} = \sqrt{\frac{5}{8}}(3x^2 - 1)$$

③ Let $f(x) = a_1 u_1 + a_2 u_2 + a_3 u_3$.

Note that $a_1 = \langle f(x), u_1\rangle = \langle 1 + 2x + 3x^2, \frac{1}{\sqrt{2}}\rangle = 2\sqrt{2}$

$$a_2 = \langle f(x), u_2\rangle = \int_{-1}^{1} (1 + 2x + 3x^2)\left(\sqrt{\frac{3}{2}}x\right) dx = \frac{2\sqrt{6}}{3}$$

$$a_3 = \langle f(x), u_3\rangle = \frac{2\sqrt{10}}{5}$$

④ $$f_1(x) = \underset{\overset{\|}{0}}{\langle f(x), u_1\rangle} u_1 + \underset{\overset{\|}{\frac{\sqrt{6}}{5}}}{\langle f(x), u_2\rangle} u_2 + \underset{\overset{\|}{0}}{\langle f(x), u_3\rangle} u_3 = \frac{3}{5}x$$

Thm 6.7 Let n-dimensional inner product space V have an orthonormal set $S = \{v_1, \dots, v_k\}$.

Then

(a) S can be extended to an orthonormal basis $\{v_1, v_2, \dots, v_k, v_{k+1}, v_{k+2}, \dots, v_n\}$ for V.

(b) If $W = \text{span}(S)$ then $\{v_{k+1}, v_{k+2}, \dots, v_n\}$ is an orthonormal basis for $W^\perp$.

(c) If W is any subspace of V, then

$V = W \oplus W^\perp$ and hence $\dim V = \dim W + \dim W^\perp$.

direct sum

see p22 for definition i.e. ① $W \perp W^\perp =$ ② $V = W + W^\perp$

also see p57 Ex. 29(b)

pf: (sketch)

(a) S can be extended to an basis

$\alpha = \{v_1, v_2, \dots, v_k, \alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_n\}$ for V.

Use the Gram-schmidt process to replace α by an orthonormal basis as desired.

(b) $W^\perp = \{x \in V \; \langle x, y \rangle = 0 \text{ for } \forall y \in S\}$

$x \in W^\perp \Rightarrow x = \sum_{i=1}^{n} a_i v_i$ for some a_i

$\Rightarrow a_1 = \langle x, v_1 \rangle = 0, \; a_2 = \langle x, v_2 \rangle = 0, \dots, a_k = 0$

$\Rightarrow x \in \text{span}(\{v_{k+1}, \dots, v_n\})$

QED

How to Compute orthogonal complement

Ex p352 Let $W = \text{span}(\{e_1, e_3\})$ be a subspace of F^3. Find $W^\perp$ and $\dim W^\perp$.

Sol:

(Method 1)

$$W^\perp = \{ v \in F^3 : \langle v, w \rangle = 0 \text{ for } \forall w \in W \}$$

$$= \{ v \in F^3 : \langle v, e_1 \rangle = 0 \text{ and } \langle v, e_2 \rangle = 0 \}$$

$$= \{ (v_1, v_2, v_3) \in F^3 : v_1 = 0 \text{ and } v_2 = 0 \}$$

$$= \{ (0, 0, *) : * \in F \}$$

$$= \text{span}(\{e_3\}).$$

So $\dim W^\perp = 1$.

(Method 2): Thm 6.7 $\Rightarrow \dim V = \dim W + \dim W^\perp$

$\Rightarrow \dim W^\perp = 1$

$\{e_1, e_2\}$ and $\{e_1, e_2, e_3\}$ are orthonormal bases for W and V, resp.

$\xRightarrow{\text{Thm 6.7(c) P352}}$ $\{e_3\}$ is a basis for $W^\perp$

QED