The adjoint of T

<u>Thm 6.8</u> Let V be an inner product space over F, $\dim V < \infty$ and $g \in L(V, F)$.

Then $\exists!\, y \in V$ s.t. $g(v) = \langle v, y \rangle$ for $\forall v \in V$.

<u>pf:</u> Let $\{v_1, v_2, \cdots, v_n\}$ be an orthonormal basis for V.

Let $y = \sum_{i=1}^{n} \overline{g(v_i)}\, v_i$.

For $v \in V$, say $v = \sum_{i=1}^{n} a_i v_i$, we have

$$\langle v, y \rangle = \langle \sum_{i=1}^{n} a_i v_i, \sum_{i=1}^{n} \overline{g(v_i)}\, v_i \rangle$$

$$= \sum_{i=1}^{n} a_i\, g(v_i) = g\left(\sum_{i=1}^{n} a_i v_i\right) = g(v)$$

Suppose $g(v) = \langle v, y' \rangle$ for $\forall v \in V$.

Then we have $\langle v, y - y' \rangle = 0 \quad \forall v \in V$.

So $\langle y - y', y - y' \rangle = 0$ and hence $y = y'$.

QED

z^*

T^*

<u>Thm 6.9</u> P358 Let V be a finite-dimensional inner product space over $\mathbb{F}$. Let $T \in L(V)$.

Then $\exists!\ T^* : V \to V$ s.t.

$\langle Tx, y \rangle = \langle x, T^*y \rangle$ for $\forall x, y \in V$.

is called the adjoint of T

Moreover, such T^* is linear.

<u>pf:</u> For $y \in V$, define $g \in L(V, \mathbb{F})$ s.t.

$g(x) = \langle Tx, y \rangle$.

Thm 6.8 $\Rightarrow \exists!\ y' \in V$ s.t. $g(x) = \langle x, y' \rangle$ for $\forall x \in V$.

Define $T^* : V \to V$ s.t. $T^*y = y'$.

Then ① $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for $\forall x, y \in V$

② $T^* \in L(V)$: Because, for $\forall x \in V$

$$\begin{aligned}\langle x, cu' + v' \rangle &= \bar{c}\langle x, u' \rangle + \langle x, v' \rangle \\ &= \bar{c}\langle Tx, u \rangle + \langle Tx, v \rangle \\ &= \langle Tx, cu + v \rangle \\ &= \langle x, (cu+v)' \rangle.\end{aligned}$$

Suppose $U \in L(V)$ s.t. $\langle Tx, y \rangle = \langle x, Uy \rangle$ for $\forall x, y \in V$.

Therefore $T^*y = Uy$ for $\forall y$. (why?)

QED

How to Compute Adjoints

<u>Thm 6.10</u> Let V be a finite-dimensional inner product space with an orthonormal basis $\beta = \{v_1, v_2, \dots, v_n\}$. If $T \in L(V)$ then

$$[T^*]_\beta = [T]_\beta^*$$

<u>pf</u>: Let $B = [T^*]_\beta$. Then $T^*(v_j) = \sum_{t=1}^{n} B_{tj} v_t$

Let $A = [T]_\beta$. Then $T(v_i) = \sum_{s=1}^{n} A_{si} v_s$.

$$
\begin{aligned}
B_{ij} &= \left\langle \sum_{t=1}^{n} B_{tj} v_t ,\ v_i \right\rangle \\
&= \overline{\langle v_i, T^* v_j \rangle} \\
&= \overline{\langle T v_i, v_j \rangle} \\
&= \overline{\left\langle \sum_{s=1}^{n} A_{si} v_s, v_j \right\rangle} \\
&= \overline{A_{ji}}
\end{aligned}
$$

So $B = A^*$.

QED

<u>Ex 2</u> p359 Let $T \in L(\mathbb{C}^2)$ s.t. $T(a_1, a_2) = (2ia_1 + 3a_2, a_1 - a_2)$. Let β be the standard ordered basis for $\mathbb{C}^2$. Compute T^*.

<u>Sol:</u> $[T]_\beta = \begin{pmatrix} 2i & 3 \\ 1 & -1 \end{pmatrix} \Rightarrow [T^*]_\beta = \begin{pmatrix} -2i & 1 \\ 3 & -1 \end{pmatrix}$

$$T^*(a_1, a_2) = [T^*]_\beta \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$= (-2ia_1 + a_2, 3a_1 - a_2).$$

END.

<u>Thm 6.11</u> $T, U \in L(V)$. V is an inner product space.

(a) $(T+U)^* = T^* + U^*$

(b) $(cT)^* = \bar{c}\, T^*$ for any $c \in F$.

(c) $(TU)^* = U^* T^*$

(d) $T^{**} = T$

(e) $I^* = I$

<u>pf</u> (c): For any $x, y \in V$,

$\langle x, U^*T^*y \rangle = \langle Ux, T^*y \rangle = \langle T(U(x)), y \rangle$
$= \langle TU(x), y \rangle$. So $(TU)^* = U^*T^*$.

Normal Operators (I)

Lemma P369 Suppose $T \in L(V)$ ← finite-dimensional inner pro. space. and $A \in M_{n\times n}(F)$.

① T has an eigenvalue $\lambda \Rightarrow T^*$ has an eigenvalue $\bar{\lambda}$.

② $\lambda \in ev(A) \Longrightarrow \bar{\lambda} \in ev(A^*)$

pf: ① Suppose $Tv = \lambda v$ for some $v \neq 0$, $\lambda \in F$.

Let $W = R(T^* - \bar{\lambda} I)$. Note that $W \oplus W^\perp = V$,

and $\dim W + \dim N(T^* - \bar{\lambda} I) = \dim V$.

So $\dim W^\perp = \dim \{y : (T^* - \bar{\lambda} I)(y) = 0\}$.

To show $v \in W^\perp$. for any $x \in V$,

$$\langle v, (T^* - \bar{\lambda} I)(x) \rangle$$

$$= \langle v, (T - \lambda I)^*(x) \rangle$$

$$= \langle (T - \lambda I)(v), x \rangle = \langle 0, x \rangle = 0$$

Therefore $\dim W^\perp \geq 1$.

② Consider L_A and L_{A^*}.

QED

Least Squares Approximation

Notation: For $x, y \in F^n$,

column vectors

$\langle x, y \rangle_n \overset{\text{def}}{=\!=} y^* x$, the standard inner product of x and y in F^n.

Lemma A Let $A \in M_{m\times n}(F)$.

Then $\text{rank}(A^*A) = \text{rank}(A)$.

Pf: $\text{rank}(A^*A)$

$= \text{column rank}(A^*A)$

$= \dim\{A^*Ax : x \in F^n\}$

$= \text{rank}(L_{A^*A}) = n - \text{nullity}(L_{A^*A})$

$= n - \dim\{x \in F^n : A^*Ax = 0\}$

Note that if $A^*Ax = 0$ for $x \in F^n$

then $x^*A^*Ax = 0$

see p360

i.e. $(Ax)^*Ax = 0$

and hence $Ax = 0$ why?

$= n - \dim\{x \in F^n : Ax = 0\}$

$= n - \text{nullity}(L_A) = \text{rank}(L_A) = \text{rank}(A)$

p362 "$m \geq n$"

Thm 6.12 Let $A \in M_{m \times n}(F)$ & $y \in F^m$.

① $\exists\ x_0 \in F^n$ s.t. $(A^*A)x_0 = A^*y$

and $\| Ax_0 - y \| \leq \| Ax - y \|$ for $\forall x \in F^n$.

② In ①, if $\operatorname{rank}(A) = n$ then $x_0 = (A^*A)^{-1}A^*y$.

pf: Let $W = \{Ax : x \in F^n\}$ be a subspace of F^m.

Let Ax_0 be the orthogonal projection of y on W.

Then $\| y - Ax \| \geq \| y - Ax_0 \|$ for $\forall x \in F^n$.

And $(y - Ax_0) \in W^\perp$

$\Rightarrow \langle Ax, y - Ax_0 \rangle_m = 0 \quad \forall x \in F^n$

p362 Lemma 1

$\Rightarrow \langle x, A^*(y - Ax_0) \rangle_n = 0 \quad \forall x \in F^n$

$\Rightarrow A^*(y - Ax_0) = 0 \Rightarrow A^*Ax_0 = A^*y$.

Moreover if $\operatorname{rank}(A) = n$, by Lemma A,

A^*A is invertible and hence

$$x_0 = (A^*A)^{-1}A^*y.$$

QED

Minimal Solutions to Systems of Linear Equ.

Def: A solution s to $Ax=b$ is called a minimal solution if $\|s\| \le \|u\|$ for all other solutions u.

Thm 6.13 (p364) Suppose $\underset{M_{m\times n}(F)}{A}x=\underset{F^m}{b}$ is consistent.

Then ⓐ $\exists!$ minimal solution s of $Ax=b$ and $s \in R(L_{A^*})$

ⓑ in ⓐ, s is the only solution to $Ax=b$ that lies in $R(L_{A^*})$.

ⓒ in ⓐ, if u satisfies $(AA^*)u=b$, then $s=A^*u$.

Pf: ⓐ Suppose $Ax=b$. Let $W=R(L_{A^*})$

Thm6.6 p350 $\Rightarrow x = s+y$ for some $s\in W$ and $y \in W^\perp$.

$\Rightarrow As = Ax - Ay = Ax = b$ since $W^\perp = N(L_A)$.

Suppose $Av=b$. To show $\|v\|^2 \ge \|s\|^2$.

$\|v\|^2 = \|(v-s)+s\|^2 = \|s\|^2 + \|v-s\|^2 + 2\,\mathrm{Re}\langle v-s, s\rangle$

$= \|s\|^2 + \|v-s\|^2 \ge \|s\|^2$

$\because v-s \in N(L_A) = W^\perp$ and $s\in W$

pf (continued)

(uniqueness) In above argument, if $\|u\|^2=\|s\|^2$

then $\|u-s\|^2=0$ and hence $u-s=0$.

ⓑ Suppose $Au=b$ and $u\in W$.

Then $\begin{cases} u-s\in W \\ A(u-s)=0 \Rightarrow u-s\in N(L_A)=W^\perp \end{cases}$

$\Rightarrow u-s\in W\cap W^\perp \Rightarrow u-s=0$

ⓒ $\left[A^*u\in R(L_{A^*})\right]$ & $\left[A(A^*u)=b\right]$

$\Rightarrow s=A^*u$ ($\because$ by ⓑ)

EX3 P364 Let $A=\begin{pmatrix}1&2&1\\1&-1&2\\1&5&0\end{pmatrix}$, $b=\begin{pmatrix}4\\-11\\19\end{pmatrix}$.

Find the minimal solution to the system $Ax=b$.

Sol: **note** $\left(A \overset{ero}{\sim} \begin{pmatrix}1&0&\frac{5}{3}\\0&1&-\frac{1}{3}\\0&0&0\end{pmatrix} \Rightarrow \text{rank}(A)=2 \right)$

Sol: Solve $(AA^*)x = b$ to get

one solution

$$x = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

Thm 6.13 says $s = A^* x = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix}$

is the minimal solution to the given system.

END

P366. Ex12

Let $W = R(L_{A^*})$, $A \in M_{m\times n}(F)$.

Show $W^\perp = N(L_A)$.

pf: $w \in W \Rightarrow w = A^* y$ for some $y \in F^m$

$z \in N(L_A) \Rightarrow Az = 0$

"$W^\perp \supseteq N(L_A)$": $\langle z, w\rangle = \langle z, A^* y\rangle$ for some y

$= \langle Az, y\rangle = \langle 0, y\rangle = 0$

• "$W^\perp \subseteq N(L_A)$": $a \in W^\perp \Rightarrow \langle a, w\rangle = 0 \ \forall w \in W$

$\Rightarrow \langle a, A^* y\rangle = 0, \forall y \in F^m$

$\Rightarrow \langle Aa, y\rangle = 0, \forall y \in F^m$

$\Rightarrow \|Aa\| = 0 \Rightarrow Aa = 0$

$\Rightarrow a \in N(L_A)$.

P337. Ex10

Suppose $\langle x, y\rangle = 0$.

Show $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.