Schur's Thm (I)

Thm 6.14 $T \in \mathcal{L}(V)$, where V is an inner product space over F with $\dim V = n$.

If $P_T(x)$ splits over F then $\exists$ orthonormal basis β for V s.t. $[T]_\beta$ is **Upper** triangular.

Pf (sketch): By induction on n. (Base) $n=1$.

Lemma (p369) + $P_T(x)$ splits over F

$\Rightarrow$ T^* has an eigenvalue λ, say $T^* v = \lambda v$ and $\|v\| = 1$.

Let $W = \text{span}(\{v\})$. Note $\dim W \geq 1$.

Claim: $W^\perp$ is T-invariant.

Pf: If $y \in W^\perp$ and $x \in W$. Then

$$\langle Ty, x\rangle = \langle Ty, cv\rangle = \langle y, cT^*v\rangle = \langle y, c\lambda v\rangle = 0$$

QED

Thm 5.21 (p314) $\Rightarrow$ $P_{T_{W^\perp}}(x) \mid P_T(x)$ $\Rightarrow$ $P_{T_{W^\perp}}(x)$ splits over F (*)

$V = W \oplus W^\perp$ (p352, Thm 6.7) and $\dim W \geq 1$

$\Rightarrow$ $\dim W^\perp < n$ (**)

Induction hypothesis $\Rightarrow$ $\exists$ orthonormal basis γ of $W^\perp$ s.t. $[T_{W^\perp}]_\gamma$ is upper triangular.

Let $\beta = \gamma \cup \{v\}$ Then $[T]_\beta = \left[\begin{array}{c|c} [T_{W^\perp}]_\gamma & * \\ \hline 0 \cdots 0 & * \end{array}\right]$ is upper triangular

QED

Schur's Thm (II)

Def: $A \in M_{n\times n}[\mathbb{F}]$ ($\mathbb{F} = \mathbb{C}$) is an Unitary Matrix if $A^*A = I$.

Thm (Thm 6.14, Thm 6.21) (Schur's Thm)

$A \in M_{n\times n}[\mathbb{F}]$ ($\mathbb{F} = \mathbb{C}$) $\Longrightarrow$ $\exists$ unitary matrix U s.t.

$$U^*AU = \begin{bmatrix} \lambda_1 & & \blacktriangle \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

an upper triangular matrix

pf: Prove by induction on n.

[Base] If $n=2$. Suppose $Ax = \lambda_1 x$, $x^*x = 1$

Gram-schmidt process $\Rightarrow \exists\, y \in \mathbb{F}^2$ s.t.

$U \overset{\text{def}}{=} [x, y]$ is unitary.

Note that

$$AU = [Ax, Ay] = [\lambda_1 x, \ b_{12}x + b_{22}y]$$

$$= [x, y]\begin{bmatrix} \lambda_1 & b_{12} \\ 0 & b_{22} \end{bmatrix} = U\begin{bmatrix} \lambda_1 & b_{12} \\ 0 & b_{22} \end{bmatrix}$$

[Induction Part] If $n = N+1$. Suppose $\lambda_1 \in ev(A)$ with $Ax = \lambda_1 x$ and $x^* x = 1$.

Gram-Schmidt process $\Rightarrow \exists\, y_1, \cdots, y_N \in \mathbb{F}^n$ s.t. $U_1 = [x, y_1, y_2, \cdots, y_N]$ is unitary.

Note that

$$U_1^* A U_1 = \left[\begin{array}{c|c} \lambda_1 & C_N \\ \hline 0 & \\ \vdots & B_N \\ 0 & \end{array}\right].$$

Suppose $ev(B_N) = \{\lambda_2, \lambda_3, \cdots, \lambda_{N+1}\}$.

Inductive hypothesis $\Rightarrow \exists$ unitary matrix U_N s.t.

$$U_N^* B_N U_N = \begin{bmatrix} \lambda_2 & & \\ & \lambda_3 & \\ 0 & & \ddots \; \lambda_{N+1} \end{bmatrix}$$

Let

$$U_{N+1} = \left[\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & U_N \\ 0 & \end{array}\right].$$

We have $(U_1 U_{N+1})^* A (U_1 U_{N+1})$

$$= U_{N+1}^* U_1^* A U_1 U_{N+1}$$

$$= U_{N+1}^* \left[\begin{array}{c|c} \lambda_1 & C_N \\ \hline 0 & \\ \vdots & B_N \\ 0 & \end{array}\right] U_{N+1}$$

$$= \left[\begin{array}{c|c} \lambda_1 & \mathbb{F} \\ \hline 0 & \\ \vdots & T_N^* B_N T_N \\ 0 & \end{array}\right] = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_{N+1} \end{bmatrix}$$

QED

Fact 1: $A \in M_{n\times n}(\mathbb{C})$. $U \in M_{n\times n}(\mathbb{C})$ is unitary.

If $U^*AU = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}$ is upper triangular.

then $ev(A) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$

Pf: $p_A(x) = \det(A - xI)$

$= \det(A - xUU^*) \quad (\because U^{-1} = U^*)$

$= (\det U)\det(U^*AU - xI)(\det U^*)$

$= \det(U^*U)\det\begin{bmatrix} \lambda_1 - x & & \\ & \lambda_2 - x & \\ 0 & & \ddots \\ & & & \lambda_n - x \end{bmatrix}$

$= (\lambda_1 - x)(\lambda_2 - x)\cdots(\lambda_n - x)$

QED

Fact 2: $A \in M_{n\times n}(\mathbb{C})$. $B \in M_{n\times n}(\mathbb{R})$.

① If A is unitary i.e. $A^*A = I$ then $|\det A| = 1$.

② If B is orthogonal i.e. $B^tB = I$ then $\det B = \pm 1$.

Fact 3 Suppose $A \in M_{n\times n}(\mathbb{C})$ with $ev(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Then

(1) $\text{trace}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

(2) $\det(A) = \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n$.

pf: Schur's Thm $\Rightarrow$ $\exists$ unitary matrix $U \in M_{n\times n}(\mathbb{C})$

s.t. $A = U \begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*$

Therefore

$$\text{trace}(A) = \text{trace}\left(U^* U \begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{bmatrix} \right.$$

$$\overset{\binom{p97}{Ex13}}{=} \text{trace}\left(\begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{bmatrix} \right) = \lambda_1 + \cdots + \lambda_n.$$

$$\det(A) = (\det U) \left(\det \begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{bmatrix} \right) (\det U^*)$$

$$= \det(U^* U) \det \begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$= \lambda_1 \lambda_2 \cdots \lambda_n. \quad (\because U^* U = I)$$

QED

Normal Operators (II)

Def: ① $T \in L(V)$, V is an inner product space.

T is normal $\overset{def}{\Longleftrightarrow}$ $TT^* = T^*T$

② $A \in M_{n\times n}(\underset{\mathbb{C} \text{ or } \mathbb{R}}{F})$.

A is normal $\overset{def}{\Longleftrightarrow}$ $AA^* = A^*A$

Remark: T is normal $\Leftrightarrow$ $[T]_\beta$ is normal
where β is an orthonormal basis for V.

Thm 6.15 P371 V: inner product space. $T \in L(V)$
and $TT^* = T^*T$. Then

(a) $\|Tv\| = \|T^*v\| \quad \forall v \in V$

(b) $Tv = \lambda v \Leftrightarrow T^*v = \bar{\lambda} v$

(c) Eigenvectors associated with distinct eigenvalues of T are orthogonal.

Pf: (b) $\langle (T-\lambda I)v, (T-\lambda I)v \rangle$

$= \langle v, (T^* - \bar{\lambda} I)((T-\lambda I)v) \rangle$

$= \langle v, (T-\lambda I)((T^* - \bar{\lambda} I)v) \rangle$ why?

$= \langle (T^* - \bar{\lambda} I)v, (T^* - \bar{\lambda} I)v \rangle$

Pf (continued)

(c) Suppose $\lambda_1, \lambda_2 \in ev(T)$, $\lambda_1 \neq \lambda_2$ and

$Tx = \lambda_1 x$, $x \neq 0$; $Ty = \lambda_2 y$, $y \neq 0$

$$\lambda_1 \langle x, y\rangle = \langle \lambda_1 x, y\rangle = \langle Tx, y\rangle = \langle x, T^* y\rangle$$

$$= \langle x, \bar{\lambda}_2 y\rangle = \lambda_2 \langle x, y\rangle$$

QED

Thm 6.16 V: inner product space over $\mathbb{C}$, $\dim V = n$

$T \in L(V)$. Then T is normal $\Leftrightarrow$ $\exists$ orthonormal basis for V consisting of eigenvectors of T

Pf: "$\Rightarrow$" ($P_T(x)$ splits over $\mathbb{C}$) + (Schur's Thm)

(Sketch) $\Rightarrow$ $\exists$ orthonormal ordered basis $\beta = \{v_1, \ldots, v_n\}$ for V s.t. $[T]_\beta$ is upper triangular.

Claim: If $\{v_1, \cdots, v_{k-1}\}$ are eigenvectors of T then v_k is an eigenvector of T.

Pf: suppose $Tv_i = \lambda v_i$ $i = 1, 2, \ldots, k-1$.

$[T]_\beta$ is upper triangular $\Longrightarrow$ $Tv_k = \sum_{i=1}^{k} c_i v_i$

$\Rightarrow$ For $j \neq k$, $c_j = \langle Tv_k, v_j\rangle$

$$= \langle v_k, T^* v_j\rangle$$
$$= \langle v_k, \bar{\lambda}_j v_j\rangle$$
$$= \lambda_j \langle v_k, v_j\rangle = 0$$

QED

Hermitian Operators

Def $T \in L(V)$, V is an inner product space over F.
$A \in M_{n\times n}(F)$. $F = \mathbb{C}$ or $\mathbb{R}$.

① T is Hermitian (self-adjoint) if $T = T^*$.

② A is Hermitian (self-adjoint) if $A = A^*$.

Lemma P373 $T \in L(V)$, $\dim V = n$. V: inner product space over F ($= \mathbb{C}$ or $\mathbb{R}$).

If $T = T^*$ then

(a) Every eigenvalue of T is real, and

(b) if $F = \mathbb{R}$ then $P_T(x)$ splits over F.

pf: (a) Method 1: Using Thm 6.15(c) P371.

Method 2: Suppose $Tv = \lambda v$, $v \neq 0$.

$$\lambda\langle v, v\rangle = \langle Tv, v\rangle = \langle v, T^* v\rangle = \langle v, Tv\rangle = \bar{\lambda}\langle v, v\rangle$$

so $\lambda = \bar{\lambda}$ i.e. $\lambda \in \mathbb{R}$. note: (a) 並沒有保証 T有 eigenvalues over F.

(b) Obviously!

Thm Let V be an inner product space over $\mathbb{R}$ and $T \in L(V)$. Then $T = T^* \Longleftrightarrow \exists$ an orthonormal basis β for V consisting of eigenvectors of T.

pf: "$\Rightarrow$"

$$T = T^*$$

$$\xRightarrow[\text{Lemma P373}]{} P_T(x) \text{ splits over } \mathbb{R}$$

$$\xRightarrow[\substack{\text{Schur's thm} \\ \text{Thm 6.14}}]{} \exists \text{ orthonormal basis } \beta \text{ for } V \text{ s.t. } [T]_\beta \text{ is upper triangular.}$$

$$\Longrightarrow [T]_\beta \text{ is a diagonal matrix.}$$

$$\left(\because ([T]_\beta)^* \underset{\text{Thm 6.10}}{=\!=} [T^*]_\beta \underset{T=T^*}{=} [T]_\beta \right)$$

$$\Longrightarrow \beta \text{ consists of eigenvectors of } T!$$

QED

Ex: Let $A = \begin{pmatrix} i & i \\ i & 1 \end{pmatrix}$. Then

① A is symmetric

② A is not normal.