Unitary Operators

- In the following let V be a finite-dimensional inner product space over $\mathbb{F}$.

Def: $T \in L(V)$.

① If $\|T(x)\| = \|x\|$ for $\forall x \in V$ and $\mathbb{F} = \mathbb{C}$, then T is called a unitary operator.

② If $\|T(x)\| = \|x\|$ for $\forall x \in V$ and $\mathbb{F} = \mathbb{R}$ then T is called an orthogonal operator.

Fact: Let $\lambda \in ev(T)$. If T is unitary or orthogonal then $|\lambda| = 1$.

pf: $\lambda \in ev(T) \Rightarrow \exists\, v \neq 0$ s.t. $Tv = \lambda v$.

$$\|Tv\| \overset{2}{=} \langle Tv, Tv \rangle$$
$$= \langle \lambda v, \lambda v \rangle = \lambda \bar{\lambda} \langle v, v \rangle \overset{\cdot}{=} \|v\|^2 |\lambda|^2$$

so $|\lambda| \overset{2}{=} 1$.

QED

Thm 6.18 P380 $T \in L(V)$, $\dim V < \infty$. ← inner product space

Then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e)

(a) $TT^* = T^*T = I$

(b) $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in V$.

(c) β is an orthonormal basis for V

$\Rightarrow T(\beta)$ is an orthonormal basis for V

(d) $\exists$ an orthonormal basis β for V s.t.

$T(\beta)$ is an orthonormal basis for V.

(e) $\|Tx\| = \|x\|$ for $x \in V$

pf: (a)$\Rightarrow$(b) $\langle Tx, Ty \rangle = \langle x, T^*Ty \rangle = \langle x, Iy \rangle = \langle x, y \rangle$

(b)$\Rightarrow$(c) let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V.

Then $\langle Tv_i, Tv_j \rangle = \langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

(c)$\Rightarrow$(d) Obviously!

Pf (continued)

(d) $\Rightarrow$ (e) Let $\beta = \{v_1, v_2, \dots, v_n\}$. Suppose $x = \sum_{i=1}^{n} a_i v_i$.

$$\|Tx\|^2 = \|\sum_{i=1}^{n} a_i T v_i\|^2$$

$$= \sum_{i=1}^{n} |a_i|^2 \quad (\because T(\beta) \text{ is an orthonormal set}).$$

$$= \sum_{i=1}^{n} |a_i|^2 \|v_i\|^2$$

$$= \|\sum_{i=1}^{n} a_i v_i\|^2 = \|x\|^2.$$

(e) $\Rightarrow$ (a)

Let $U = T^*T - I$. Then $U = U^*$, and hence ($\because$ Thm 6.16 + Thm 6.17)

$\exists$ orthonormal basis $\beta = \{v_1, v_2, \dots, v_n\}$ s.t.

$$U v_i = \lambda_i v_i \quad i = 1, 2, \dots, n.$$

Consider $\bar{\lambda}_i \langle v_i, v_i \rangle = \langle v_i, \lambda_i v_i \rangle$

$$= \langle v_i, U v_i \rangle$$

$$= \langle v_i, (T^*T - I) v_i \rangle$$

$$= \langle v_i, T^*T v_i \rangle - \langle v_i, v_i \rangle$$

$$= \langle T v_i, T v_i \rangle - \langle v_i, v_i \rangle \overset{\because (e)}{=} 0, \text{ and}$$

hence $\lambda_i = 0, \forall i$.

For any $x = \sum_{i=1}^{n} a_i v_i$, we have

$$U(x) = \sum_{i=1}^{n} a_i U v_i = \sum_{i=1}^{n} a_i \lambda_i v_i = 0 \quad \text{i.e.}$$

$T^*T - I = T_0$ and hence $T^*T = I$. see p.107, Ex10 to $TT^* = I$. QED

Fact p381 suppose $F = \mathbb{R}$.

T is Hermitian & orthogonal $\iff$ $\exists$ an orthonormal
$T^* = T$ $\quad$ $\|v\| = \|Tv\| \ \forall v$

basis β for V consisting of eigenvectors of T

with corresponding eigenvalues of absolute value 1.

Pf: "$\Rightarrow$" $(T^* = T) + (F = \mathbb{R}) + (\text{Thm } 6.17)$

$\Longrightarrow$ $\exists$ orthonormal basis $\{v_1, \dots, v_n\}$ for V

s.t. $Tv_i = \lambda_i v_i$ for $\forall i$.

$\Rightarrow |\lambda_i|^2 \langle v_i, v_i \rangle = \lambda_i \bar{\lambda}_i \langle v_i, v_i \rangle = \langle Tv_i, Tv_i \rangle = \|Tv_i\|^2$

So $|\lambda_i| = 1$ $\quad\quad\quad\quad \underset{T \text{ is orthogonal}}{=} \|v_i\|^2$

"$\Leftarrow$" V has an orthonormal basis $\beta = \{v_1, \dots, v_n\}$ s.t.

$Tv_i = \lambda_i v_i$, $|\lambda_i| = 1$, $\lambda_i \in \mathbb{R}$, $\forall i$.

$\Rightarrow [T]_\beta = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$ and hence $[T]_\beta = [T]_\beta^*$.

$\Rightarrow$ T is Hermitian

Pf (continued)

For any $x = \sum_{i=1}^{n} a_i v_i$,

$$T^*T(x) = T^*\left(\sum_{i=1}^{n} a_i \lambda_i v_i\right)$$

$$= \sum_{i=1}^{n} a_i \lambda_i T^* v_i$$

$$= \sum_{i=1}^{n} a_i |\lambda_i|^2 v_i$$

$\because |\lambda_i| = 1$

$$= \sum_{i=1}^{n} a_i v_i = x$$

$\left(\begin{array}{c} T \text{ is Hermitian} \\ \Downarrow \\ \because T \text{ is normal}, \\ \Downarrow \\ T^* v_i = \bar{\lambda}_i v_i,\ \forall i \end{array}\right)$

So $T^*T = I$ and hence $T^*T = TT^* = I$.

(Thm 6.18) + ($TT^* = T^*T = I$)

$\Rightarrow \|Tx\| = \|x\|$ for $\forall x \in V$

$\Rightarrow T$ is orthogonal ($\because F = \mathbb{R}$)

QED.

Unitary Matrix

Def: $A \in M_{n\times n}[F]$, $B \in M_{n\times n}[F]$. (F: $\mathbb{C}$ or $\mathbb{R}$)

- A is called an orthogonal matrix if $A^tA = I$.
- A is called a unitary matrix if $A^*A = I$.
- A is orthogonally equivalent to B if ($A \overset{oe}{\sim} B$)
  $\exists$ orthogonal matrix Q s.t. $A = Q^{-1}BQ$.
- A is unitarily equivalent to B if ($A \overset{ue}{\sim} B$)
  $\exists$ unitary matrix Q s.t. $A = Q^{-1}BQ$.

Thm 6.19 P384 $A \in M_{n\times n}[\mathbb{C}]$. Then

A is normal $\Longleftrightarrow$ $A \overset{ue}{\sim}$ a diagonal matrix D

i.e. $\exists$ unitary matrix Q s.t. $A = Q^{-1}DQ$.

Pf: "$\Rightarrow$" Consider $L_A \in L(\mathbb{C}^n)$. L_A is normal.

Thm 6.16 $\Rightarrow$ $\exists$ orthonormal basis $\beta = \{v_1, v_2, \dots, v_n\}$

for $\mathbb{C}^n$ (inner product space over $\mathbb{C}$) s.t. $Av_i = \lambda_i v_i$ $i = 1, 2, \dots, n$.

$$\Rightarrow [L_A]_\beta = \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{D} \qquad \lambda_i \in \mathbb{C}$$

Pf (continued)

So $[I L_A I]_\beta^\beta = D$

$\Rightarrow [I]_\gamma^\beta [L_A]_\gamma^\gamma [I]_\beta^\gamma = D$ ← standard ordered basis for $\mathbb{C}^n$: so $\langle u,v\rangle = v^* u$

$\Rightarrow Q^{-1} A \underbrace{[v_1, v_2, \ldots, v_n]}_{Q} = D$

Therefore $A = QDQ^{-1}$. Q is unitary! $(\because Q^*Q = \begin{bmatrix} v_1^* \\ v_2^* \\ v_n^* \end{bmatrix} [v_1\, v_2 \cdots v_n] = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix})$

"$\Leftarrow$" $A \overset{ue}{\sim} D$

$\Rightarrow A = Q^{-1} D Q$ for some unitary matrix Q

$\Rightarrow A = Q^* D Q \; (\because Q^* Q = I)$

$\Rightarrow A^* A = Q^* D^* Q Q^* D Q$

$= Q^* D^* D Q \; (\because QQ^* = I)$

$= Q^* D D^* Q$ why?

$= Q^* D Q Q^* D^* Q$

$= AA^*$

QED

Thm 6.20 $A \in M_{n\times n}(\mathbb{R})$. Then

A is symmetric $\Longleftrightarrow$ $A \underset{\text{o.e.}}{\sim}$ a diagonal matrix D

pf: Consider $L_A \in \mathcal{L}(\mathbb{R}^n)$. ← inner product space over $\mathbb{R}$, so $\langle u, v\rangle = v^t u$

Then $L_A = L_A^*$ (check it!).

Thm 6.17 $\Longrightarrow$ $\exists$ orthonormal basis $\beta = \{v_1, \dots v_n\}$ for $\mathbb{R}^n$ s.t. $Av_i = \lambda_i v_i$, $i = 1, 2, \dots n$. $\lambda_i \in \mathbb{R}$.

$$\Longrightarrow [L_A]_\beta = \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}}_{D}$$

$$\Longrightarrow [I]^\beta_\gamma [L_A]^\gamma_\gamma \underbrace{[I]^\gamma_\beta}_{Q = [v_1, v_2 \dots v_n]} = D$$

$$\Longrightarrow Q^{-1} A Q = D$$

$\Longrightarrow A = QDQ^{-1}$ and

Q is orthogonal. $\left(\because Q^t Q = \begin{bmatrix} v_1^t \\ v_2^t \\ \vdots \\ v_n^t \end{bmatrix} [v_1, \dots, v_n] = [\langle v_i, v_j\rangle]_{n\times n} = I\right)$

QED