Generalized Eigenspaces of T

Def: $T \in L(V)$ & $\lambda \in ev(T)$.

The set $K_\lambda^T = \{x \in V : (T-\lambda I)^p(x) = 0 \text{ for some } p \in \mathbb{N}\}$ is called the generalized eigenspace of T corresponding to λ. A nonzero vector $x \in K_\lambda^T$ is called a generalized eigenvector of T corresponding to λ.

Thm 7.1 p485 $T \in L(V)$ and $\lambda \in ev(T)$.

Then ① K_λ^T is a subspace of V & $E_\lambda^T \subseteq K_\lambda^T$.

② K_λ^T is T-invariant.

③ $\mu \neq \lambda \Rightarrow (T - \mu I) : K_\lambda^T \longrightarrow K_\lambda^T$ is 1-to-1

i.e. $ev(T_{K_\lambda^T}) = \{\lambda\}$

pf: (sketch)

① $x, y \in K_\lambda^T,\ c \in F \Rightarrow (T-\lambda I)^p x = 0,\ (T-\lambda I)^q y = 0$

$\Rightarrow (T-\lambda I)^{p+q}(x+cy) = 0$

② $x \in K_\lambda^T \Rightarrow (T-\lambda I)^p x = 0$

$\Rightarrow (T-\lambda I)^p(Tx) = T((T-\lambda I)^p(x)) = T0 = 0$

$\Rightarrow Tx \in K_\lambda^T$

③ $x \in K_\lambda^T \setminus \{0\}$ & $(T-\mu I)x = 0 \Rightarrow \exists\, p$ s.t. $(T-\lambda I)^p x = 0,\ (T-\lambda I)^{p-1} x \neq 0$

$\Rightarrow (T-\lambda I)^{p-1} x \in E_\lambda^T \cap E_\mu^T = \{0\}$ $\left(\because (T-\mu I)(T-\lambda I)^{p-1}x = (T-\lambda I)^{p-1}(T-\mu I)x\right)$ a contradiction

QED

Thm 7.2 $T \in L(V)$, V is over $\mathbb{F}$, $\dim V < \infty$, $P_T(x)$ splits over $\mathbb{F}$. Then

$\lambda \in ev(T)$ and $am(\lambda) = m$ $\Longrightarrow$ ① $\dim K_\lambda^T \leq m$ ③ $K_\lambda^T = N((T-\lambda I)^m)$

② $P_{T_W}(x) = (-1)^{\dim W}(x-\lambda)^{\dim W}$, where $W = K_\lambda^T$.

algebraic multiplicity P264

Pf: Let $W = K_\lambda^T$

①+② (W is T-invariant) + Thm 5.21 P314

$\Rightarrow \quad P_{T_W}(x) \mid P_T(x)$

$\Rightarrow \quad (-1)^{\dim W}(x-\lambda)^{\dim W} \mid P_T(x) \quad (\because ev(T_W) = \{\lambda\}$ by Thm 7.1 $)$

$\Rightarrow \quad \dim W \leq m$

③ $\quad x \in W$

$\Longrightarrow \quad P_{T_W}(T_W)(x) = 0 \quad (\because$ Cayley-Hamiltonian Thm says that $P_{T_W}(T_W) = 0$ $)$

zero trans.

$\Longrightarrow (-1)^{\dim W}(T_W - \lambda I_W)^{\dim W}(x) = 0 \quad (\because ②)$

$\Longrightarrow \quad (T_W - \lambda I_W)^m(x) = 0 \quad (\because \dim W \leq m)$

$\Longrightarrow \quad (T-\lambda I)^m(x) = 0$

$\Longrightarrow \quad x \in N((T-\lambda I)^m)$

QED

Thm 7.3 $T\in L(V)$, V is over F, $\dim V = n$.

$$\underbrace{\{\lambda_1, \cdots, \lambda_k\}}_{\text{distinct}} = ev(T) \Rightarrow V = \{u_1+u_2+\cdots+u_k : u_i \in K_{\lambda_i}^T, 1\le i\le k\}$$

i.e. $V = K_{\lambda_1}^T + K_{\lambda_2}^T + \cdots + K_{\lambda_k}^T$

pf: By induction on k.

$$k=1 \Rightarrow P_T(x) = (-1)^n(x-\lambda_1)^n \xRightarrow{\text{Cayley–Hamiltonian Thm}} (-1)^n(T-\lambda_1 I)^n = T_0$$

$$\Rightarrow V = K_{\lambda_1}^T$$

Let m = algebraic multiplicity (λ_k). $W = R((T-\lambda_k I)^m)$

Suppose $P_T(x) = (x-\lambda_k)^m g(x)$.

Claim: W is T-invariant.

pf: $v\in W \Rightarrow v = (T-\lambda_k I)^m x$ for some $x\in V$

$$\Rightarrow Tv = T(T-\lambda_k I)^m x = (T-\lambda_k I)^m (Tx) \in W. \blacksquare$$

Claim: For $i<k$, $T-\lambda_k I : K_{\lambda_i}^T \to K_{\lambda_i}^T$ is a bijection

pf: Thm 7.1(b) $\Rightarrow$ $T-\lambda_k I$ is 1-to-1 from $K_{\lambda_i}^T$ to $K_{\lambda_i}^T$.

$\Rightarrow$ $T-\lambda_k I$ is onto from $K_{\lambda_i}^T$ to $K_{\lambda_i}^T$

($\because$ Thm 2.5 p71)

Claim*: For $i<k$, $(T-\lambda_k I)^m : K_{\lambda_i}^T \to K_{\lambda_i}^T$ is **onto**.

Claim$^\Delta$: For $i<k$, we have $K_{\lambda_i}^T \subseteq W$ & $\lambda_i \in ev(T_W)$.

pf:

$$E_{\lambda_i}^T \subseteq K_{\lambda_i}^T \underset{\text{by claim*}}{\subseteq} W \Rightarrow \exists\, x\in W\setminus\{0\} \text{ s.t. } Tx = \lambda_i x$$

$$\Rightarrow \exists\, x\in W\setminus\{0\} \text{ s.t. } T_W x = \lambda_i x$$

$$\Rightarrow \lambda_i \in ev(T_W)$$

pf (continued)

Claim: $\lambda_K \notin ev(T_W)$

pf: Assume $\exists\, v \in W \setminus \{0\}$ s.t. $T_W v = \lambda_K v$

$\Rightarrow \exists\, y \in V$ s.t. $v = (T - \lambda_K I)^m y$ and hence

$$(T - \lambda_K I)^{m+1} y = (T_W - \lambda_K I)(v) = 0$$

$\Rightarrow y \in K_{\lambda_K}^{T} = N((T - \lambda_K I))^m$ ($\because$ Thm 7.2)

$\Rightarrow v = (T - \lambda_K I)^m y = 0$ a contradiction.

Claim: $ev(T_W) = \{\lambda_1, \lambda_2, \cdots, \lambda_{K-1}\}$

pf: (W is T-invariant) + Thm 5.21 P314

$\Rightarrow ev(T_W) \subseteq ev(T) = \{\lambda_1, \dots, \lambda_K\}$

$\Rightarrow ev(T_W) = \{\lambda_1, \dots, \lambda_{K-1}\}$ ($\because \lambda_i \in ev(T_W)$ $i = 1, 2, \dots, K-1$ by claim)
($\because \lambda_K \notin ev(T_W)$ by above claim)

Claim $x \in V \Rightarrow x = \sum_{i=1}^{K} v_i$ for some $v_i \in K_{\lambda_i}^{T}$ $1 \le i \le K$.

pf: $(T - \lambda_K I)^m x \in W$ (by the def. of W)

$\Rightarrow (T - \lambda_K I)^m x = w_1 + w_2 + \dots + w_{K-1}$ for some $w_i \in K_{\lambda_i}^{T_W}$ $1 \le i \le K-1$

$\left(\begin{array}{l} \because \text{induction hypothesis} + T_W \in L(W) + ev(T_W) = \{\lambda_1, \dots, \lambda_{K-1}\} \\ \text{imply } W = \{w_1 + \dots + w_{K-1} : w_i \in K_{\lambda_i}^{T_W},\ 1 \le i \le K-1\} \end{array}\right)$

$\Rightarrow (T - \lambda_K I)^m x = \sum_{i=1}^{K-1} (T - \lambda_K I)^m v_i$ for some $v_i \in K_{\lambda_i}^{T}$ $1 \le i \le K-1$

$\left(\because (T - \lambda_K I)^m : K_{\lambda_i}^{T} \to K_{\lambda_i}^{T} \text{ is onto, and } w_i \in K_{\lambda_i}^{T_W} \subseteq K_{\lambda_i}^{T}\right)$

$\Rightarrow (T - \lambda_K I)^m (x - \sum_{i=1}^{K-1} v_i) = 0$ and hence $x - \sum_{i=1}^{K-1} v_i = v_K$ for some $v_K \in K_{\lambda_K}^{T}$.

$\Rightarrow x = \sum_{i=1}^{K} v_i$ for some $v_i \in K_{\lambda_i}^{T}$ $1 \le i \le K$.

QED

Thm 7.4 P487

$T \in L(V)$, V is over F, $\dim V < \infty$ with

$$\mathrm{Spec}(T) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix}.$$

Let β_i be an ordered basis for $K_{\lambda_i}^T$, $1 \le i \le k$.

Then (a) $\beta_i \cap \beta_j = \phi$ when $i \neq j$

(b) $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered basis for V.

(c) $\dim(K_{\lambda_i}^T) = m_i$, $1 \le i \le k$.

note: (b) $\Rightarrow$ $V = K_{\lambda_1}^T \oplus \cdots \oplus K_{\lambda_k}^T$

Pf: (a)

Assume $x \in \beta_i \cap \beta_j \subseteq K_{\lambda_i}^T \cap K_{\lambda_j}^T$.

$x \in K_{\lambda_i}^T \setminus \{0\}$

$\Rightarrow (T - \lambda_j I)x \in K_{\lambda_i}^T \setminus \{0\}$ $(\because (T-\lambda_j I) : K_{\lambda_i}^T \to K_{\lambda_i}^T$ is 1-to-1$)$

$\Rightarrow (T - \lambda_j I)^2 x \in K_{\lambda_i}^T \setminus \{0\}$ (same reason)

$\vdots$

$\Rightarrow (T - \lambda_j I)^p x \neq 0$ for any $p \in \mathbb{N}$

$\Rightarrow x \notin K_{\lambda_j}^T$ a contradiction.

(b) $V = K_{\lambda_1}^T + K_{\lambda_2}^T + \cdots + K_{\lambda_k}^T$ $(\because$ Thm 7.3 $)$

note: In fact we can use (b) to prove (a) !

$$\Rightarrow \begin{cases} V = \mathrm{span}(\beta) \\ \dim V \le |\beta| \le \sum_{i=1}^{k} |\beta_i| \underset{\substack{\uparrow \\ \text{Thm 7.2 ①}}}{\le} \sum_{i=1}^{k} m_i = \dim V \end{cases}$$

$\Rightarrow \beta$ is an ordered basis for V.

(c) In the proof of (b), we shown $|\beta_i| = m_i$ $1 \le i \le k$.

Corollary P488 $T \in L(V)$, V is over $\mathbb{F}$, $\dim V < \infty$, $P_T(x)$ splits over $\mathbb{F}$. Then T is diagonalizable $\Leftrightarrow E_\lambda^T = K_\lambda^T$ for $\forall \lambda \in ev(T)$.

pf: Let $Spec(T) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & & m_k \end{pmatrix}$.

"$\Rightarrow$"

$(T$ is diagonalizable$) + (\text{Thm } 5.9) \Rightarrow gm(\lambda_i) = am(\lambda_i) \ \forall i$.

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$(E_{\lambda_i}^T \subseteq K_{\lambda_i}^T) + (\text{Thm } 7.4) \Rightarrow am(\lambda_i) = \dim K_{\lambda_i}^T \geq \underbrace{\dim E_{\lambda_i}^T}_{= gm(\lambda_i)}$

Therefore $E_{\lambda_i}^T = K_{\lambda_i}^T$. $\forall i$.

"$\Leftarrow$"

$(E_{\lambda_i}^T = K_{\lambda_i}^T \ \ \forall i) + (\text{Thm } 7.3) \Rightarrow V = E_{\lambda_1}^T \oplus \cdots \oplus E_{\lambda_k}^T$

Therefore T is diagonalizable ($\because$ Thm 5.11 or Thm 5.9)

Def: $T \in L(V)$, $x \in K_\lambda^T \setminus \{0\}$ and $\lambda \in ev(T)$.

generalized eigenvector

Suppose $p = \min\{m \in \mathbb{N} : (T - \lambda I)^m x = 0\}$.

The ordered set

$$\gamma = \{ \underbrace{(T-\lambda I)^{p-1} x}_{\text{initial vector}}, (T-\lambda I)^{p-2} x, \ldots, (T-\lambda I)x, \underbrace{x}_{\text{end vector}} \}$$

is called a **cycle of generalized eigenvectors of T corresponding to λ.**

Note: The above cycle has **length** p.

Ex $T \in L(V)$, $x \in K_\lambda^T \setminus \{0\}$ and $\lambda \in ev(T)$.

Show that the cycle of generalized eigenvectors of T corresponding to λ

$$\gamma = \{ (T-\lambda I)^{p-1} x, (T-\lambda I)^{p-2} x, \dots, (T-\lambda I)x, x \}$$

is a l. independent set.

i.e. Every cycle of generalized eigenvectors of a linear operator is l. ind.

Pf (sketch: by example) Let $p=3$

Suppose

$$a_2 (T-\lambda I)^2 x + a_1 (T-\lambda I) x + a_0 x = 0$$

Then we have

$$(T-\lambda I)^2 a_0 x = -a_2 (T-\lambda I)^4 x - a_1 (T-\lambda I)^3 x$$

$$= 0 - 0 = 0, \text{ and hence } \underset{\substack{\uparrow \\ \text{scalar}}}{a_0 = 0}$$

since $(T-\lambda I)^2 x \neq \underset{\substack{\uparrow \\ \text{zero vector}}}{0}$.

By the sam way, we arrive at $a_0 = a_1 = a_2 = 0$.

QED

Note 1: $\gamma \cap$ eigenvectors$(T) = \{ (T-\lambda I)^{p-1} x \}$

Pf: For $k < p-1$, $(T-\lambda I)^k x \in K_\lambda^T \setminus \{0\}$.

Suppose $\mu \in ev(T)$ and $\mu \neq \lambda$.

Thm 7.1 $\Rightarrow (T-\mu I)\left[(T-\lambda I)^k x \right] \neq 0$

$\Rightarrow (T-\lambda I)^k x$ is not an eigenvector of T.

Thm 7.5 p489 $T \in L(V)$, V is over F, $\dim V < \infty$,

$P_T(x)$ splits over F.

Suppose $\gamma_i = \{ (T-\lambda_i I)^{p-1} x, \cdots, (T-\lambda_i I)x, x \}$

are disjoint cycles of generalized eigenvectors

of T corresponding to eigenvalues λ_i, $1 \le i \le k$.

If $\beta = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is an ordered basis for V

then

(a 1) $W_i = \text{span}(\gamma_i)$ is T-invariant for all i

(a2) $[T_{W_i}]_{\gamma_i}$ has the form $\begin{bmatrix} \lambda_i & 1 & & O \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ O & & & \lambda_i \end{bmatrix}$

(b) $[T]_\beta = \begin{pmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_k \end{pmatrix}$ s.t. $A_i = [T_{W_i}]_{\gamma_i}$.

pf: **(a1)** let $v_j = (T-\lambda_i I)^j x$, $0 \le j \le p-1$.

$(T-\lambda_i I)^0 = I$

For $0 \le j < p-1$: $T(v_j) = v_{j+1} + \lambda_i v_j \in W_i$

$T(v_{p-1}) = 0 + \lambda_i v_{p-1} \in W_i$

Therefore $T(W_i) \subseteq W_i$

(a2) $T_W(v_{p-1}) = \lambda_i v_{p-1}$

$T_W(v_{p-2}) = v_{p-1} + \lambda_i v_{p-2}$

$T_W(v_{p-3}) = v_{p-2} + \lambda_i v_{p-3}$

$\vdots$

$T_W(v_1) = v_2 + \lambda_i v_1$

$T_W(v_0) = v_1 + \lambda_i v_0$

so $$[T_{W_i}]_{\gamma_i} = \begin{array}{c} v_{p-1} \\ v_{p-2} \\ v_{p-3} \\ \vdots \\ \\ v_0 \end{array} \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & \lambda_i & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ \vdots & \vdots & \vdots & & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}$$

(a3) Easy!

Thm 7.6

p489 $T \in L(V)$, $\lambda \in ev(T)$, $\dim V < \infty$. $K_\lambda^T \setminus \{0\}$

Suppose $\gamma_i = \{ \underbrace{(T-\lambda I)^{p_i} x_i}_{y_i}, (T-\lambda I)^{p_i - 1} x_{i}, \ldots, (T-\lambda I) x_i, x_i \}$ $1 \le i \le g$ are cycles of generalized eigenvectors of T corresponding to λ s.t.

$\{y_1, y_2, \cdots, y_g\}$ is a l. ind. set.

(and hence $y_1, y_2, \ldots, y_g$ are distinct vectors)

Then we have

(a) $\gamma_i \cap \gamma_j = \phi$ for any $1 \le i < j \le g$.

(b) The set $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_g$ is l. ind.

pf: (b) By induction on $|\gamma|$. Suppose $|\gamma| = n$.

(sketch) Let $W = \text{span}(\gamma)$ and $U = (T-\lambda I)_W$. ($\because$ W is $T-\lambda I$ invariant)

let $\gamma_i' = \gamma_i \setminus \{x_i\}$, $1 \le i \le g$ and $\gamma' = \bigcup_i \gamma_i'$.

Claim γ' is a basis for $R(U)$.

pf: Clearly, $R(U) = \text{Span}(\gamma')$.

moreover, induction hypothesis $\Rightarrow$ γ' is l. ind.

QED

Note that

$$n = |\gamma| \ge \dim W = \underbrace{\dim(R(U))}_{\substack{\| \\ |\gamma'| = n-g}} + \underbrace{\dim(N(U))}_{\substack{\vee\!\!\| \\ g}} \ge n$$

($\because$ l. ind. set $\{y_1, \ldots, y_g\} \subseteq N(U)$)

Therefore $\dim W = n$. Together with the fact $W = \text{span}(\gamma)$, $|\gamma| = n$, we have γ is l. ind.

QED

Thm 7.7 If $T \in L(V)$, $\dim V < \infty$ and $\lambda \in ev(T)$

then K_λ^T has an ordered basis $\beta = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_q$

where

(1) $\gamma_1, \gamma_2, \ldots, \gamma_q$ are cycles of generalized eigenvectors corresponding to λ.

(2) $\gamma_i \cap \gamma_j = \phi$ for all $1 \le i < j \le q$.