How to find Jordan Forms.

EX2 Let $A=\begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$.

① Find a Jordan canonical form J for A.

② Find a matrix $Q \in M_{3\times 3}(\mathbb{R})$ whose columns are the vectors in an ordered basis β for V s.t. $Q^{-1}AQ = J$.

Sol: (sketch)

$$P_A(x) = -(x-3)(x-2)^2$$

$$\operatorname{spec}(A) = \begin{array}{c} \\ \\ \\ \end{array}\!\!\begin{array}{cc} \lambda_1 & \lambda_2 \\ \left(\begin{matrix} 3 \\ 1 \end{matrix}\right. & \left.\begin{matrix} 2 \\ 2 \end{matrix}\right) \\ m_1 & m_2 \end{array}$$

Note that $K_{\lambda_1}^{A} = N((A-\lambda_1 I)^{m_1}) = N((A-3I))$

$$= \left\{ t\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

Then $\beta_1 = \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$ is an ordered basis for $K_{\lambda_1}^{A}$ consisting of disjoint cycles of generalized eigenvectors of A corresponding to $\lambda_1 = 1$.

Sol (continued)

Note that $K^A_{\lambda_2} = N((A-\lambda_2 I)\ ) = N((A-2I)^2)$

$$= \left\{ a\begin{pmatrix}-1\\2\\0\end{pmatrix} + b\begin{pmatrix}1\\0\\2\end{pmatrix} : a, b \in \mathbb{R} \right\}$$

Let $v_1 = \begin{pmatrix}-1\\2\\0\end{pmatrix}$. Then $\beta_2 = \{ (A-2I)v_1, v_1 \} = \left\{ \begin{pmatrix}1\\-3\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\\0\end{pmatrix} \right\}$

is an ordered basis for $K^A_{\lambda_2}$ consisting of disjoint cycles of generalized eigenvectors of A corresponding to $\lambda_2 = 2$.

Note that $\mathbb{R}^3 = K^A_{\lambda_1} \oplus K^A_{\lambda_2}$ and has an ordered basis $\beta = \beta_1 \cup \beta_2$.

Let $Q = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -3 & 2 \\ 1 & -1 & 0 \end{bmatrix}$

Then $Q^{-1}AQ = \underbrace{\left[\begin{array}{c|cc} 3 & 0 & 0 \\ \hline 0 & 2 & 1 \\ 0 & 0 & 2 \end{array}\right]}_{J}$ is a Jordan canonical form for A.

END

EX3 Let $A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$.

① Find a Jordan canonical form J for A.

② Find a matrix $Q \in M_{3\times 3}(\mathbb{R})$ whose columns are the vectors in an ordered basis β for V s.t. $Q^{-1}AQ = J$.

Sol: $P_A(x) = -(x+1)^3 \Longrightarrow \operatorname{spec}(A) = \begin{pmatrix} \lambda_1 \\ -1 \\ 3 \end{pmatrix}$

Note that $K_{\lambda_1}^{A} = N((A-\lambda_1 I)^{m_1}) = N((A+I)^{\overset{m_1}{3}}) = \mathbb{R}$.

Choose $v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. （請試試看為何選 $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ 時有問題！）

Then
$$\beta_2 = \left\{ (A-\lambda_1 I)^2 v_1,\ (A-\lambda_1 I)v_1,\ v_1 \right\} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is an ordered basis for $K_{\lambda_1}^{A}$ consisting of a cycle of generalized eigenvectors of A corresponding to $\lambda_1 = -1$. Let $Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Then
$$Q^{-1}AQ = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

END

Ex p500 let $A=\begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix} \in M_{4\times 4}(\mathbb{R})$.

① Find a Jordan canonical form J for A.

② Find a matrix $Q \in M_{4\times 4}(\mathbb{R})$ whose columns are the vectors in an ordered basis β for $\mathbb{R}^4$ s.t. $Q^{-1}AQ=J$

Sol: (sketch) $P_A(x)=(x-3)(x-2)^3$

$$\mathrm{Spec}(A)=\begin{array}{c} \begin{matrix} \lambda_1 & \lambda_2 \end{matrix} \\ \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} \\ \begin{matrix} m_1 & m_2 \end{matrix} \end{array}$$

Note that $K_{\lambda_1}^{A} = N((A-\lambda_1 I)^{m_1}) = N((A-2I)^3)$

$$=\left\{ a\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\2\\0\end{pmatrix} + c\begin{pmatrix}0\\1\\0\\2\end{pmatrix} : a,b,c \in \mathbb{R} \right\}$$

Choose $v_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$.

Then $\gamma_{11}=\{(A-\lambda_1 I)^{m_1}v_1, \dots, (A-\lambda_1 I)^0 v_1\} = \left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\}$

Choose $v_2=\begin{pmatrix}0\\1\\2\\0\end{pmatrix}$.

Then $\gamma_{12}=\{(A-\lambda_1 I)^{m_1}v_2, \dots (A-\lambda_1 I)^0 v_2\} = \left\{\begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\2\\0\end{pmatrix}\right\}$

Sol: (continued)

Note that $\beta_1 = \gamma_{11} \cup \gamma_{12} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right\}$

is an ordered basis for $K_{\lambda_1}^A$ consisting of disjoint cycles of generalized eigenvectors of A corresponding to $\lambda_1 = 2$.

Next, we have $K_{\lambda_2}^A = N((A-\lambda_2 I)^{m_2}) = N((A-3I))$

$$= \left\{ a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} : a \in \mathbb{R} \right\}.$$

Choose $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

Then $\gamma_{21} = \{ (A-\lambda_2 I)^{m_2} v_3, \ldots, (A-\lambda_2 I)^0 v_3 \}$

$$= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Note that $\beta_2 = \gamma_{21}$ is an ordered basis for $K_{\lambda_2}^A$ consisting of a cycle of generalized eigenvector of A corresponding to $\lambda_2 = 3$.

Sol: (continued)

Let $Q = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$.

Then we have

$$Q^{-1}AQ = \left(\begin{array}{c|cc|c} 2 & 0 & 0 & 0 \\ \hline 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 3 \end{array}\right).$$

END

EX p506 let $A = \begin{pmatrix} 0&1&0&0&0&0 \\ 0&0&0&2&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$.

(a) Find a Jordan canonical form J for A

(b) Find a matrix $Q \in M_{6\times 6}(\mathbb{R})$ s.t.

$$Q^{-1}AQ = J.$$

Sol: (sketch)

$$P_A(x) = x^6 \Longrightarrow \operatorname{spec}(A) = \begin{pmatrix} \overset{\lambda_1}{0} \\ \underset{m_1}{6} \end{pmatrix}$$

$$K_{\lambda_1}^{A} = N\left((A-\lambda_1 I)^{m_1}\right) = N(A^6) = \mathbb{R}^6.$$

- Choose $u_1 = \begin{pmatrix} 0\\0\\0\\1\\0\\0 \end{pmatrix}$.

Then $\beta_1 = \{(A-\lambda_1 I)^{m_1} u_1, \dots, (A-\lambda_1 I)^0 u_1\} = \left\{ \begin{pmatrix} 2\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\2\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1\\0\\0 \end{pmatrix} \right\}$

- Choose $u_2 = \begin{pmatrix} 0\\0\\0\\0\\1\\0 \end{pmatrix}$.

Then $\beta_2 = \{(A-\lambda_1 I)^{m_1} u_2, \dots, (A-\lambda_1 I)^0 u_2\} = \left\{ \begin{pmatrix} 0\\0\\0\\0\\1\\0 \end{pmatrix} \right\}$

- Choose $u_3 = \begin{pmatrix} 0\\0\\0\\0\\0\\1 \end{pmatrix}$

Then $\beta_3 = \{(A-\lambda_1 I)^{m_1} u_3, \dots, (A-\lambda_1 I)^0 u_3\} = \left\{ \begin{pmatrix} 0\\0\\1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\0\\0\\1 \end{pmatrix} \right\}$

Sol: (continued)

Let $Q = \begin{pmatrix} 2 & 6 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Then

$$Q^{-1}AQ = \left(\begin{array}{ccc|c|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

END