Minimal Polynomial

p517

- let $\dim V < \infty$ and V is over the field F.
  let $T \in L(V)$ and $A \in M_{n \times n}(F)$.

Def: (1) A polynomial over F $p(t)$ is called a minimal polynomial of T (A) if $p(t)$ is a monic (leading coefficient is 1) polynomial of least positive degree for which $p(T)=0$. ($p(A)=0$)

- let β be an ordered basis for V.

Notation: 本課程使用 $m_T(t)$、$m_A(t)$ 來各別表示 T 與 A 的 minimal polynomial.

Thm 7.13 $m_T(t) = m_{[T]_\beta}(t)$.

Note: The characteristic polynomial

$$P_T(t) \stackrel{\text{def}}{=} P_{[T]_\beta}(t)$$

Thm 7.12

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zero transformation

(1) If $h(x) \in P(F)$ and $h(T) = 0$

then $m_T(x) \mid h(x)$ (i.e. $m_T(x)$ divides $h(x)$)

(2) The minimal polynomial of T is unique.

Pf: (1) Suppose $h(x) = m_T(x) q(x) + r(x)$

such that $\deg r(x) < \deg m_T(x)$.

If $r(x)$ is not a zero polynomial then

$$h(T) = m_T(T) q(T) + r(T)$$

$\Rightarrow \quad 0 = 0\, q(T) + r(T)$

$\Rightarrow \quad r(T)$ is a zero transformation.

a contradiction!

(2) By (1) and the property that $m_T(x)$ is monic.

QED

Thm 7.14 p517 $\lambda \in \mathbb{F}$

(1) $\lambda \in ev(T) \Leftrightarrow m_T(\lambda) = 0$

(2) $P_T(\lambda) = 0 \Leftrightarrow m_T(\lambda) = 0$

pf:

(2) "$\Leftarrow$"

$$m_T(\lambda) = 0 \Rightarrow m_T(x) = (x-\lambda)\, q(x)$$

$$\Rightarrow q(T) \neq 0$$ ← zero transformation

why?

$$\Rightarrow \exists\, v \in V \text{ s.t. } q(T)v \neq 0$$ ← zero vector

Let $u = q(T)v$.

Note that
$$Tu - \lambda u = (T-\lambda I)u$$
$$= (T-\lambda I)\, q(T)v$$
$$= m_T(T)v$$
$$= 0$$ ← zero vector

"$\Rightarrow$"

$$\lambda \in ev(T)$$
$$\Rightarrow \exists\, v \neq 0 \text{ s.t. } Tv = \lambda v$$
$$\Rightarrow m_T(T)v = m_T(\lambda)v$$
$$\Rightarrow 0 = m_T(\lambda)v$$

↑ zero vector

$$\Rightarrow m_T(\lambda) = 0$$ ← value $(\because v \neq 0)$

Corollary:

If $P_T(x) = (\lambda_1 - x)^{n_1}(\lambda_2 - x)^{n_2} \cdots (\lambda_k - x)^{n_k}$ and

$\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct

then $m_T(x) = (\lambda_1 - x)^{m_1}(\lambda_2 - x)^{m_2} \cdots (\lambda_k - x)^{m_k}$

for some integers $1 \le m_i \le n_i$ for $\forall i$.

Thm 7.16. P520

T is diagonalizable $\Longleftrightarrow$

$m_T(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_k)$ where

$\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of T

pf: "$\Rightarrow$" T is diagonalizable

$\Rightarrow \exists$ an ordered basis β for V s.t.

$[T]_\beta$ is a diagonal matrix

$u \in \beta \Rightarrow \exists\, i$ s.t. $Tu = \lambda_i u$

$\Rightarrow m_T(T)u = (T-\lambda_1 I)\cdots(T-\lambda_k I)u$

$= (T-\lambda_1 I)\cdots(T-\lambda_{i-1} I)(T-\lambda_{i+1} I)\cdots(T-\lambda_k I)$

$(T-\lambda_i I)u$

$= 0$

$\Rightarrow m_T(T)v = 0$ for any $v \in V$