Degrees and choice numbers

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Degree & Choice Numbers

$ch(G) \overset{\text{def}}{=\!=} \min\{ k : \text{for } \forall\, s: V \to \binom{\mathbb{N}}{k},\ \exists \text{ a proper vertex coloring } c \text{ of } G \text{ s.t. } c(v) \in s(v)\ \forall v \in V \}$

$col(G) \overset{\text{def}}{=\!=} \min\{ d : \forall \text{ subgraph of } G \text{ contains a vertex of degree smaller than } (<)\ d \}$

$\underline{\underline{\text{fact}}}$ the minimum d s.t. $\exists$ an acyclic orientation of G in which every outdegree is smaller than $(<)$ d.

$\underline{\underline{\text{fact}}}$ the minimum d s.t. G is $(d-1)$-degenerate.

Fact: $\chi(G) \leq ch(G) \leq col(G)$

- a graph G is $(d-1)$-degenerate if every subgraph of it has a vertex of degree at most d.

Lower bound for ch(G)

Thm let G be a simple graph with $\delta_G \geq d$. If $s \in \mathbb{N}$ and

$$d > \frac{4}{(\log_2 e)^2}(s^2+1)^2 2^{2s}$$

then $ch(G) > s$.

pf: $n \stackrel{\text{def}}{=} |V_G|$.

$S \stackrel{\text{def}}{=} \{1, 2, 3, \cdots, s^2\}$

Goal: To show $\exists\ L : V_G \to \binom{S}{s}$ such that there is no proper coloring $c : V_G \to S$ with $c(v) \in L(v)$ for every $v \in V_G$.

- Choose $B \subseteq V_G$: $\Pr(v \in B) = 1/\sqrt{d}$
- Choose $L(b) \in \binom{S}{s}$ for each $b \in B$: $\Pr(L(b) = U) = \dfrac{1}{\binom{s^2}{s}}$.
- A vertex $v \in V_G$ is good if

old probability space!

  (a) $v \in \overline{B}$

  (b) for $\forall\ T \in \binom{S}{\lceil s^2/2 \rceil}$, there exists $b \in N(v) \cap B$ s.t. $L(b) \subseteq T$.

An estimation

Fact $\binom{\lceil s^2/2 \rceil}{s} \Big/ \binom{s^2}{s} \geq \frac{1}{2^{s+1}}$, where $s \in \mathbb{N}$.

pf:

$$\text{LHS} = \frac{\lceil s^2/2 \rceil (\lceil s^2/2 \rceil - 1) \cdots (\lceil s^2/2 \rceil - s + 1)}{s^2 \, (s^2 - 1)(s^2 - 2) \cdots (s^2 - s + 1)}$$

$$\geq \frac{1}{2^s} \prod_{i=0}^{s-1} \frac{s^2 - 2i}{s^2 - i} = \frac{1}{2^s} \prod_{i=0}^{s-1} \left(1 - \frac{i}{s^2 - i}\right)$$

$$\underset{\text{why?}}{\geq} \frac{1}{2^s} \left(1 - \sum_{i=0}^{s-1} \frac{i}{s^2 - i}\right) \quad \left(s \geq 3 \Rightarrow \frac{i}{s^2 - i} < 1\right)$$

$$= \frac{1}{2^s} \left(1 - \sum_{i=0}^{s-1} \frac{i}{s^2 - s}\right) = \frac{1}{2^{s+1}}.$$

QED

pf $\Pr(u \text{ is not good}) = \Pr(\overline{(a)}) + \Pr((a) \cap \overline{(b)})$

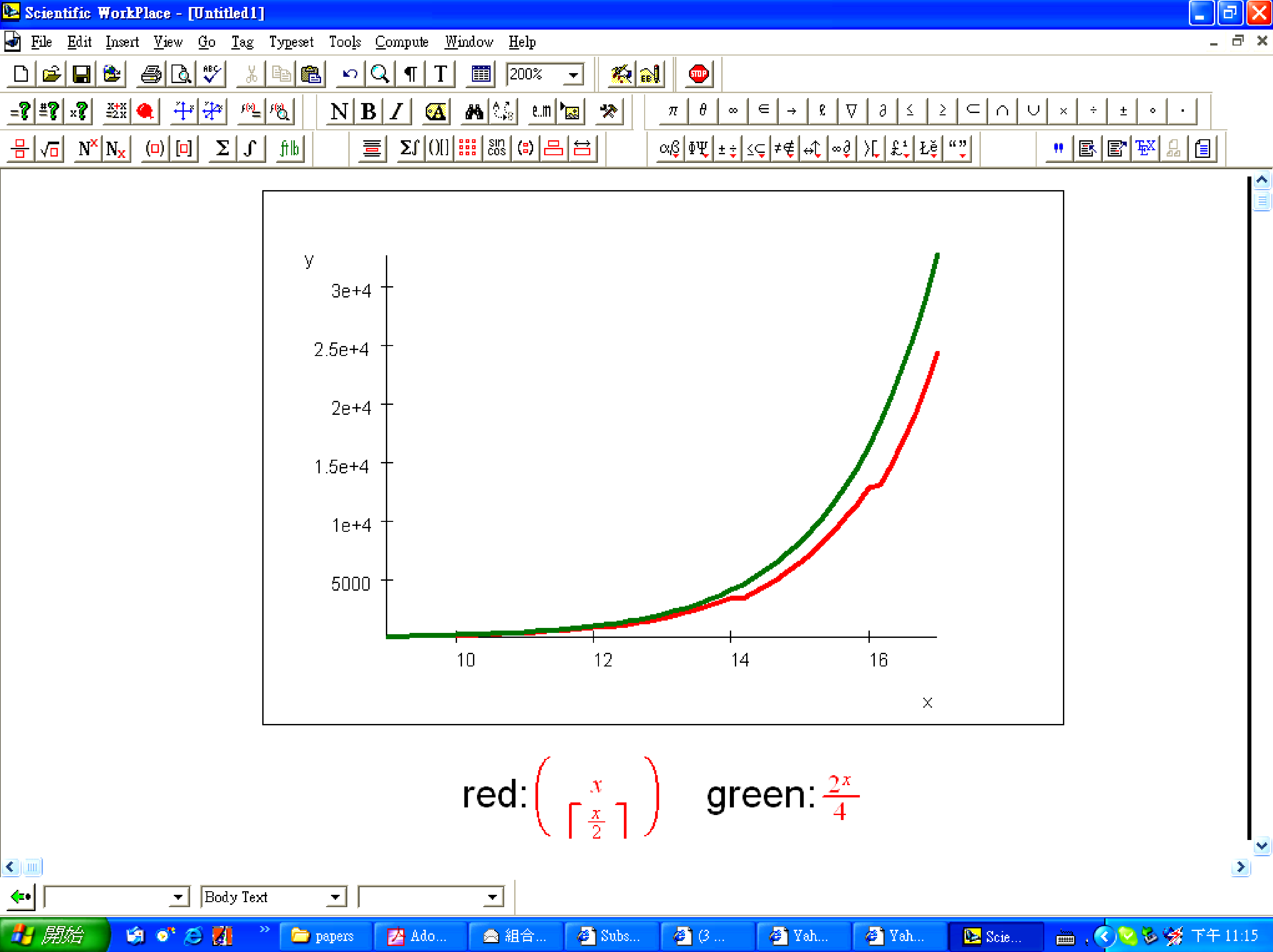
$$= \frac{1}{\sqrt{d}} + \left(1 - \frac{1}{\sqrt{d}}\right)\Pr(\overline{(b)}) \leq \frac{1}{\sqrt{d}} + \Pr(\overline{(b)}).$$

$$\Pr(\overline{(b)}) = \Pr\Big(\bigcup_{T \in \binom{S}{\lceil s^2/2 \rceil}} \bigcap_{b \in N(u)} \overline{\{b \in B \text{ and } L(b) \subseteq T\}}\Big)$$

$$\leq \sum_{T \in \binom{S}{\lceil s^2/2 \rceil}} \left(1 - \frac{1}{\sqrt{d}} \frac{\binom{\lceil s^2/2 \rceil}{s}}{\binom{s^2}{s}}\right)^d.$$

Therefore $\Pr(u \text{ is not good}) \leq \frac{1}{\sqrt{d}} + \binom{s^2}{\lceil s^2/2 \rceil}\left(1 - \frac{1}{\sqrt{d}} \frac{1}{2^{s+1}}\right)^d$

$$\leq \frac{1}{\sqrt{d}} + \frac{2^{s^2}}{4} e^{-\frac{\sqrt{d}}{2^{s+1}}} < \frac{1}{4} \text{ by hypothesis.}$$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pf **Claim** $\exists B \subseteq V_G$ and $L(b) \in \binom{S}{s}$ for each $b \in B$ s.t.

① $|B| \le 2n/\sqrt{d}$ and ② $\overline{B}$ contains $\ge \frac{n}{2}$ good vertices.

pf: $\Pr\left(|B| \le 2n/\sqrt{d} \text{ and } \sum_{v \in V_G} I_{\{v \text{ is not good}\}} \le \frac{n}{2} \right)$

$$\ge 1 - \Pr\left(\sum_{v \in V_G} I_{\{v \in B\}} \ge 2n/\sqrt{d} \right) - \Pr\left(\sum_{v \in V_G} I_{\{v \text{ is not good}\}} \ge \frac{n}{2} \right)$$

$$\ge 1 - \frac{\sum_{v \in V_G} \Pr(v \in B)}{2n/\sqrt{d}} - \frac{\sum_{v \in V_G} \Pr(v \text{ is not good})}{n/2}$$

$$> 1 - \frac{n/\sqrt{d}}{2n/\sqrt{d}} - \frac{n/4}{n/2} = 0$$

END

pf: Fix a partial coloring $c_{|B}: B \to S$ with $c_{|B}(b) \in L(b)$, $\forall b \in B$.

$A \overset{\text{def}}{=} \{ v \in \overline{B} : v \text{ is a good vertex} \}$

- Choose $L(a) \in \begin{bmatrix} S \\ s \end{bmatrix}$ for each $a \in A$: $\Pr(L(a) = U) = \dfrac{1}{\binom{s^2}{s}}$.

New probability space!

$T_a \overset{\text{def}}{=} \{ c_{|B}(b) : \quad a \sim b \in B \}$ for each $a \in A$.

Claim: $|T_a| \geq \lceil \frac{s^2}{2} \rceil$ for each $a \in A$.

pf: $T_a \cap T \neq \phi$ for each $T \in \begin{bmatrix} S \\ \lceil s^2/2 \rceil \end{bmatrix}$

implies $|T_a| \geq \lceil \frac{s^2}{2} \rceil$ since $s^2 = \lceil \frac{s^2}{2} \rceil + \lfloor \frac{s^2}{2} \rfloor$ and

$\lfloor \frac{s^2}{2} \rfloor + 1 \geq \lceil \frac{s^2}{2} \rceil$.

QED of claim.

pf <u>claim</u> For each $a \in A$, $\Pr(\text{a can be colored}) \leq 1 - \frac{1}{2^{s+1}}$.

<u>pf</u> $\Pr(\text{a can be colored}) \leq \Pr(L(a) \nsubseteq T_a)$

$$= 1 - \Pr(L(a) \subseteq T_a) = 1 - \frac{\binom{|T_a|}{s}}{\binom{s^2}{s}} \leq 1 - \frac{\binom{\lceil \frac{s^2}{2} \rceil}{s}}{\binom{s^2}{s}} \leq 1 - \frac{1}{2^{s+1}}.$$

QED of claim.

$E_{C|B} \overset{\text{def}}{=} \{ C_{|B}$ can be extended to a <u>proper</u> coloring $c: A \cup B \to S$ with $c(a) \in L(a)$ for each $a \in A \}$

$$\Pr(E_{C|B}) = \Pr\Big(\bigcap_{a \in A} \{\text{a can be colored}\}\Big)$$

$$= \prod_{a \in A} \Pr\{\text{a can be colored}\}$$

$$\leq \left(1 - \frac{1}{2^{s+1}}\right)^{|A|} \leq \left(1 - \frac{1}{2^{s+1}}\right)^{\frac{n}{2}} \leq e^{-\frac{n}{2^{s+2}}}.$$

pf $\Pr\left(\bigcap \overline{E_{C|B}}\right) = 1 - \Pr\left(\bigcup_{C|B} E_{C|B}\right) \geq 1 - \sum_{C|B} \Pr(E_{C|B})$

$C|B$ is a proper coloring from B to S.

$$\geq 1 - s^{|B|} \Pr(E_{C|B}) \geq 1 - s^{|B|} e^{-\frac{n}{2^{s+2}}}$$

$$= 1 - e^{\underbrace{|B|\ln s - \frac{n}{2^{s+2}}}_{*}} > 0.$$

Note that $* < 0 \Leftarrow |B| \ln s < \frac{n}{2^{s+2}}$

$\Leftarrow \frac{2n}{\sqrt{d}} \ln s < \frac{n}{2^{s+2}}$

$\Leftarrow 2^{2s}(64)(\ln s)^2 < d$

$\Leftarrow$ hypothesis and $s \geq 3$. $\left(\because \frac{100}{(\log_2 e)^2} \doteq 48.045,\ 16(\ln 3)^2 \doteq 19.311\right)$

QED

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