Fractional Chromatic Number

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χ_f

- Let $\mathcal{I}(G) = \{ I : I \text{ is an independent set of } G \}$.
- A fractional coloring is a mapping $c : \mathcal{I}(G) \to [0,1]$ s.t.

  for every $x \in V_G$, $\sum_{x \in I \in \mathcal{I}(G)} c(I) \geq 1$

- Fractional chromatic number

$$\chi_f(G) = \inf \left\{ \sum_{I \in \mathcal{I}(G)} c(I) : c \text{ is a fractional coloring of } G \right\}$$

Another Formula for χ_f

$$\chi_f(G) = \min \sum_i c_i$$

$$[c_1, c_2, \cdots, c_s] \begin{array}{c|c} & v_1 \quad v_2 \quad \cdots \quad v_n \\ \begin{array}{c} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_s \end{array} & \left[\quad A \quad \right] \end{array} \geq [1, 1, \cdots, 1]$$

Where $c_1, c_2, \cdots, c_s \in [0,1]$.

$$a_{ij} = \begin{cases} 1 & \text{if } v_j \in I_i \\ 0 & \text{if } v_j \notin I_i \end{cases}$$

Duality

$$\chi_f = \min c^t 1_s$$
$$c^t A \geq 1_n^t, \quad c_i \in [0,1] \ \forall i$$

$$\omega_f = \max 1_n^t \omega$$
$$A\omega \leq 1_s, \quad \omega_j \in [0,1] \ \forall j$$

$$\underline{\chi_f} = \min c^t 1_s$$
$$c^t A \geq 1_n^t, \quad c_i \geq 0 \ \forall i$$

$$\overline{\omega_f} = \max 1_n^t \omega$$
$$A\omega \leq 1_s, \quad \omega_j \geq 0 \ \forall j$$

Fact: $\omega \leq \overline{\omega_f} = \omega_f = \chi_f = \underline{\chi_f} \leq \chi.$

Fact: $\chi_f = \min c^t 1_s$

$$c^t A = 1_n^t, \quad c_i \geq 0 \ \forall i$$

pf (sketch) Suppose $\mathcal{X}_f$ has an optimal solution

$$\hat{c}^t = [\underset{I_1}{\tfrac{1}{2}}, \underset{I_2}{\tfrac{1}{4}}, \underset{I_3}{\tfrac{2}{3}}] \in \mathbb{Q}^n \quad (\because a_{ij} \in \mathbb{Z} \ \ \forall i,j)$$

Consider $$\hat{c}^t_{new} = [\underbrace{\tfrac{1}{12}, \cdots, \tfrac{1}{12}}_{6}, \underbrace{\tfrac{1}{12}, \cdots, \tfrac{1}{12}}_{3}, \underbrace{\tfrac{1}{12}, \cdots, \tfrac{1}{12}}_{8}]$$

Consider A_{new} as follows

$$\begin{array}{cc} & \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{array}{r} 6\left\{\begin{array}{c} I_1 \\ \vdots \\ I_1 \end{array}\right. \\ 3\left\{\begin{array}{c} I_2 \\ \vdots \\ I_2 \end{array}\right. \\ 8\left\{\begin{array}{c} I_3 \\ \vdots \\ I_3 \end{array}\right. \end{array} & \left[\begin{array}{cccccc} 1/12 & & 1/12 & 1/12 & & 1/12 \\ \vdots & & \vdots & \vdots & & \vdots \\ 1/12 & & 1/12 & 1/12 & & 1/12 \\ & 1/12 & 1/12 & 1/12 & & \\ & \vdots & \vdots & \vdots & & \\ & 1/12 & 1/12 & 1/12 & & \\ 1/12 & 1/12 & 1/12 & 1/12 & & 1/12 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1/12 & 1/12 & 1/12 & 1/12 & & 1/12 \end{array}\right] \\ & \begin{array}{cccccc} & & \| & \| & & \\ & & 1+\frac{2}{12} & 1+\frac{3}{12} & & \end{array} \end{array}$$

Then we **delete** row1 & row2 of A_{new}, and add two new rows as follows

$$\begin{array}{cc} & \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{array}{c} I^* \\ I^* \end{array} & \left[\begin{array}{cccccc} 1/12 & & & 1/12 & & 1/12 \\ 1/12 & & & 1/12 & & 1/12 \end{array}\right] \end{array}$$

Proceed this way to get an optimal solution $\tilde{c}$ with $\tilde{c}^t A = [1, 1, \cdots, 1]$.

QED

K-tuple coloring

K-tuple coloring $\chi_k(G) \overset{\text{def}}{=} \min\left\{n : \exists\, c : V_G \to \binom{[n]}{k} \text{ s.t. } x \sim y \Rightarrow c(x) \cap c(y) = \phi\right\}$

note: 1. $\chi_k(G) = \min\left\{n : G \xrightarrow{h} \mathbf{K}(n,k)\right\}$ 2. $\chi_1(G) = \chi(G)$

Fact $\chi_{a+b} \le \chi_a + \chi_b$

Fact $\lim_{k\to\infty} \frac{\chi_k(G)}{k} = \inf_k \frac{\chi_k(G)}{k}$

pf: Suppose $\lim_{i\to\infty} \frac{\chi_{k_i}(G)}{k_i} = \limsup_{k\to\infty} \frac{\chi_k(G)}{k}$.

Fix k. For sufficiently large $k_i = q_i k + r$, $0 \le r < k$.

$$\frac{\chi_{k_i}(G)}{k_i} \le \frac{q_i \chi_k(G) + r\chi_1(G)}{q_i k + r} \le \frac{\chi_k(G)}{k} + \frac{r\chi_1(G)}{q_i}$$

So $\limsup_{k\to\infty} \frac{\chi_k(G)}{k} \le \inf_k \frac{\chi_k(G)}{k} \le \liminf_{k\to\infty} \frac{\chi_k(G)}{k}$. QED

fractional coloring & k-tuple coloring

$\chi_f(G,k) \overset{\text{def}}{=} \min\ c^t 1$

$$[c_1, \cdots, c_\ell] \begin{array}{c} \\ I_1 \\ I_2 \\ \vdots \\ I_\ell \end{array} \begin{array}{c} v_1\, v_2 \cdots v_n \\ \left[\begin{array}{ccc} & & \\ & & \\ & & \\ & & \end{array} \right] \end{array} \geq [\overbrace{k, k, \cdots, k}^{n}], \quad c_i \in \{0, 1, 2, \cdots, k\}$$

Fact $\chi_k(G) = \chi_f(G,k) = \chi(G[K_k])$

Fact (1) $\chi_f(G) = \inf\limits_{k} \dfrac{\chi_k(G)}{k}$ (2) $\chi_f(G) = \dfrac{\chi_k(G)}{k}$ for some k.

Pf: Note that $\chi_f(G) \leq \chi_k(G)/k$ for any k.

$\chi_f(G)$ has an optimal solution $[c_1, \cdots, c_t]$ s.t. $c_i \in \mathbb{Q}$ $\forall i$, since $\chi_f(G)$ is the solution of a LP with integer coefficients.

We can reduce c_i's to a common denominator, say $c_i = \dfrac{d_i}{\hat{k}}$ $i = 1, 2, \dots, t$.

Then $\chi_f(G) \geq \dfrac{\chi_{\hat{k}}(G)}{\hat{k}}$.

QED

Another formula for χ_f

Fact $\chi_f(G) = \inf\left\{ \frac{n}{k} : \exists\, c : V_G \to \binom{[n]}{k} \text{ s.t. } x \sim y \Rightarrow c(x) \cap c(y) = \phi \right\}$

$= \inf\left\{ \frac{n}{k} : G \xrightarrow{h} \text{Kneser}(n,k) \right\}$

Def $\omega_k(G) = \max\ 1_n^t w$, $\begin{array}{c|c} & v_1\ v_2 \cdots v_n \\ \hline I_1 & \\ \vdots & \\ I_t & \end{array} \left[\quad \right] \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \le \left.\begin{bmatrix} k \\ \vdots \\ k \end{bmatrix}\right\} t$, $w_i \in \{0, 1, \ldots k\}$

Fact (1) $\omega \le \frac{\omega_k}{k} \le \omega_f = \chi_f \le \frac{\chi_k}{k} \le \chi$ for $\forall k$.

(2) $\omega_{a+b}(G) \ge \omega_a(G) + \omega_b(G)$

(3) $\lim_{k\to\infty} \frac{\omega_k(G)}{k} = \sup_k \frac{\omega_k(G)}{k}. = \omega_f(G)$

(4) $\omega_f(G) = \frac{\omega_k(G)}{k}$ for some k.

$\chi_f(G[H])$

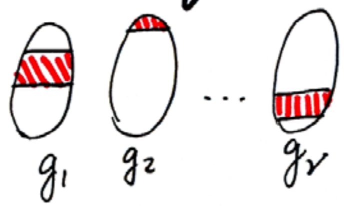
Fact $\chi_f(G[H]) = \chi_f(G)\,\chi_f(H)$.

pf: "$\le$" Let $[c_1, \ldots, c_t]$ & $[c_1', \ldots, c_{s}']$ be optimal solutions for $\chi_f(G)$ and $\chi_f(H)$ resp. To find a feasible solution for $\chi_f(G[H])$ with weight $(c_1 + \cdots + c_t)(c_1' + \cdots + c_s')$.

"$\ge$" To show $\omega_f(G[H]) \ge \omega_f(G)\,\omega_f(H)$.

Let $[w_1, \ldots, w_\nu]$ & $[w_1', \ldots, w_{\nu'}']$ be optimal solutions for $\omega_f(G)$ and $\omega_f(H)$ resp. (where $|V_G| = \nu$, $|V_H| = \nu'$)

Define $w''(g,h) = w_g w_h'$. To show w'' is a feasible solution for $\omega_f(G[H])$. Consider an independent set I of $G[H]$.



QED

$\chi(G \oplus H)$

- $E_{G\oplus H} = \{ (g,h)(g',h') : g \sim g' \text{ in } G \text{ or } h \sim h' \text{ in } H \}$

Fact $\chi_f(G)\chi(H) \leq \chi(G \oplus H)$

Pf: View1: $\chi(G\oplus H) \geq \chi(G[H]) = \chi(G[K_t])$ where $\chi(H)=t$.

$$= \frac{\chi(G[K_t])}{t}\chi(H) \geq \chi_f(G)\chi(H).$$

View2: $\chi(G\oplus H) \geq \chi(G[H])$

$$= \chi(G[K_t]) \text{ where } t=\chi(H)$$

$$\geq \chi_f(G[K_t]) = \chi_f(G)\chi_f(K_t)$$

$$= \chi_f(G)\chi(H)$$

QED

$$\inf_n \sqrt[n]{\chi(\underbrace{G\oplus\cdots\oplus G}_{n})}$$

Thm $\chi_f(G) = \inf_n \sqrt[n]{\chi(G^n)}$, where $G^n = \underbrace{G\oplus\cdots\oplus G}_{n}$

pf: "$\leq$" $\chi(G) \geq \chi_f(G)$ and $\chi(G^t) \geq \chi_f(G)\chi(G^{t-1})$ $t = 2, 3, \cdots, n$.

$\Rightarrow \sqrt[n]{\chi(G^n)} \geq \chi_f(G)$, $\forall n$

"$\geq$" $\exists\, m$ s.t. $\chi_f(G) = \chi_m(G)/m$. Say $\chi_m(G) = \sum_i c_i$ s.t.

$$[c_1, \cdots, c_t] \begin{array}{c|c|} & v_1\ v_2 \cdots v_N \\ I_1 & \\ I_2 & \\ \vdots & \\ I_t & \end{array} \geq [m, m, \cdots, m], \text{ where } c_i \in \{0, 1, 2, \cdots, m\}.$$

let $\mathcal{F} = \{\underbrace{I_1, \cdots, I_1}_{c_1}, \underbrace{I_2, \cdots, I_2}_{c_2}, \cdots\cdots, \underbrace{I_t, \cdots, I_t}_{c_t}\}$ be a multiset.

Construct L random independent sets $I_1^i \times I_2^i \times \cdots \times I_n^i$ $i = 1, 2, \cdots, L$. in G^n. by choosing I_j^i from $\mathcal{F}$ uniformly and independently. (where set $L = \lceil \{\chi_f(G)(1+\varepsilon)\}^n \rceil$)

pf (continued) So we have $Pr\{\underset{\uparrow \text{any } g\in V(G)}{g} \in I_j^i\} \geq \dfrac{m}{c_1+c_2+\cdots+c_t} = \dfrac{m}{\chi_m(G)}$.

Note that $Pr\left\{ \bigcup_{(g_1,\dots,g_n)\in G^n} \bigcap_{1\le i\le L} \{ (g_1,\dots,g_n) \notin I_1^i \times I_2^i \times \cdots \times I_n^i \} \right\}$

$$\le \sum_{(g_1,\dots,g_n)\in G^n} \prod_{i=1}^{L} Pr\{ (g_1,\dots,g_n) \notin I_1^i \times I_2^i \times \cdots \times I_n^i \}$$

$$\le \sum_{(g_1,\dots,g_n)} \prod_{1\le i\le L} \left[1 - \left(\frac{m}{\chi_m(G)}\right)^n \right] = \nu^n \left[1 - \left(\frac{m}{\chi_m(G)}\right)^n \right]^L$$

$$\le \nu^n \left[1 - \left(\frac{1}{\chi_f(G)}\right)^n \right]^{\{\chi_f(G)(1+\varepsilon)\}^n} \quad (\because \chi_f(G) > 1)$$

$$\le \nu^n \left(\frac{1}{e}\right)^{(1+\varepsilon)^n} \quad \left(\because \left(1-\frac{1}{x}\right)^x \uparrow \frac{1}{e}\right)$$

< 1 as n sufficiently large. Hence $\chi(G^n) \le L$ as $n \gg 0$.

So $\sqrt[n]{\chi(G^n)} \le \sqrt[n]{\{\chi_f(G)(1+\varepsilon)\}^{n+1}} \Rightarrow \inf_n \sqrt{\chi(G^n)} \le \overline{\lim_{n\to\infty}} [\chi_f(G)(1+\varepsilon)]^{\frac{n+1}{n}} = \chi_f(G)(1+\varepsilon)$

Therefore $\inf_n \sqrt[n]{\chi(G^n)} \le \chi_f(G)$. **QED**