Azuma's Inequality

Lecturer: Dr. Hong-Gwa Yeh
Department of Mathematics
National Central University
hgyeh@math.ncu.edu.tw

Azuma's Inequality

Thm 7.2.1 Let $Z_0, Z_1, \cdots, Z_n = X$ be a martingale w.r.t. the filter $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{F}$ on a probability space $(\Omega, \mathcal{F}, P)$.

Suppose $|Z_i - Z_{i-1}| \leq c_i$ for $i = 1, 2, \cdots, n$.

Then for any $t > 0$, we have

$$P(X - \mathcal{E}X \geq t) \leq e^{-\frac{t^2}{2\sum_{i=1}^n c_i^2}}.$$

Note: we also have $P(X - \mathcal{E}X \leq -t) \leq e^{-\frac{t^2}{2\sum_{i=1}^n c_i^2}}$.

Note: Azuma Kazuoki ≡ 吾妻一興

pf W.l.o.g assume $Z_0=0$ and $c_i=1$ $\forall i$.

First we show $\mathcal{E}(e^{uZ_i}) \le e^{\frac{iu^2}{2}}$ for $u>0$ by induction on i.

$$\mathcal{E}(e^{uZ_i}) = \mathcal{E}(\mathcal{E}(e^{uZ_{i-1}} e^{uZ_i - uZ_{i-1}} \mid \mathcal{F}_{i-1}))$$

$$= \mathcal{E}(e^{uZ_{i-1}} \mathcal{E}(e^{u(Z_i - Z_{i-1})} \mid \mathcal{F}_{i-1})) \quad (\because e^{uZ_{i-1}} \text{ is } \mathcal{F}_{i-1}\text{-measurable})$$

$$\le \mathcal{E}\left(e^{uZ_{i-1}} \mathcal{E}\left((Z_i - Z_{i-1})\left(\frac{e^u - e^{-u}}{2}\right) + \frac{e^u + e^{-u}}{2} \,\Big|\, \mathcal{F}_{i-1}\right)\right) \quad (\text{note } 1)$$

$$= \mathcal{E}\left(e^{uZ_{i-1}} \frac{e^u + e^{-u}}{2}\right) \quad (\text{note } 2)$$

$$\le e^{\frac{(i-1)u^2}{2}}\left(1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \frac{u^6}{6!} + \cdots\right) \quad \text{induction hypothesis)}$$

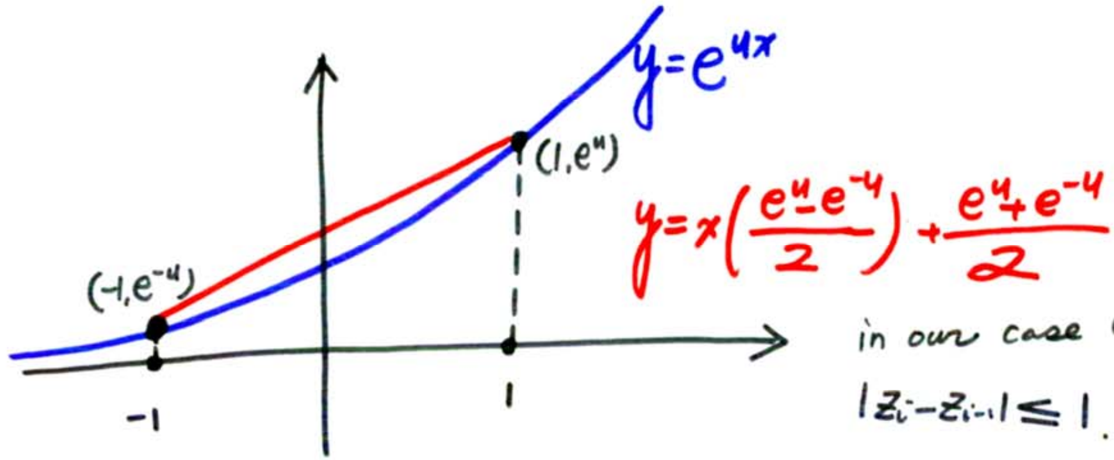
$$\le e^{\frac{(i-1)u^2}{2}}\left(1 + \left(\frac{u^2}{2}\right) + \frac{\left(\frac{u^2}{2}\right)^2}{2!} + \frac{\left(\frac{u^2}{2}\right)^3}{3!} + \cdots\right) = e^{\frac{(i-1)u^2}{2}} e^{\frac{u^2}{2}} = e^{\frac{iu^2}{2}}.$$

Thus $\mathcal{P}(X \ge t) \underset{u>0}{=} \mathcal{P}(e^{uX} \ge e^{ut}) \le \mathcal{E}e^{uX}/e^{ut} \le e^{\frac{nu^2}{2}}/e^{ut}$ $(\because X = Z_n)$

$$= e^{\frac{nu^2}{2} - ut} \le e^{-\frac{t^2}{2n}} \text{ by letting } u = \frac{t}{n}.$$

QED

Note 1



in our case we have $|z_i - z_{i-1}| \le 1$.

Note 2

we note that $\mathcal{E}(z_i - z_{i-1} \mid \mathcal{F}_{i-1}) = \mathcal{E}(z_i \mid \mathcal{F}_{i-1}) - z_{i-1} = 0$ a.s.

Example

EX Toss a fair coin n times. Count the number of heads.

Let $X_i = \begin{cases} 1 & \text{if } i\text{th toss is a head} \\ 0 & \text{o.w.} \end{cases}$. We have $X_1, \cdots, X_n \overset{iid}{\sim} B(1, \frac{1}{2})$

Let $f(X_1, X_2, \cdots, X_n) = \sum_{i=1}^n X_i$. Consider the Doob's martingale

$$Y_0 = \mathcal{E}(f(X_1, \cdots, X_n)), \cdots, Y_i = \mathcal{E}(f(X_1, \cdots, X_n) \mid X_1, \cdots, X_i), \cdots Y_n = f(X_1, \cdots, X_n).$$

Note that $|Y_i - Y_{i-1}| = |\mathcal{E}(f(\underset{\sim}{X}) \mid X_1, \cdots, X_i) - \mathcal{E}(f(\underset{\sim}{X}) \mid X_1, \cdots, X_{i-1})|$

$$\overset{X_1, \cdots X_n \text{ are iid}}{=} |(X_1 + \cdots + X_i) + \mathcal{E}(\sum_{j=i+1}^n X_j) - (X_1 + \cdots + X_{i-1}) + \mathcal{E}(\sum_{j=i}^n X_j)|$$

$$= |X_i - \mathcal{E}X_i| = |X_i - \tfrac{1}{2}| = \tfrac{1}{2}$$

Azuma's inequality $\Rightarrow P(f(\underset{\sim}{X}) - \frac{n}{2} \geq t) \leq e^{-\frac{2t^2}{n}}$.

END

Concentration of the chromatic number

Thm p96 Thm 7.2.4 [Shamir & Spencer 1987] Let $n \geq 2$ and $p \in (0,1)$. Then we have

$$\mathcal{P}\left(\left|\chi(G_{n,p}) - \mathcal{E}\chi(G_{n,p})\right| \geq t\right) \leq 2e^{-\frac{t^2}{2(n-1)}}.$$

Pf: Consider vertex exposure martingale $Z_1, Z_2, \dots, Z_n$, where $Z_i = \mathcal{E}\left(\chi(G_{n,p}) \mid I_{xy}, \{x,y\} \in \left[\binom{[i]}{2}\right]\right)$.

It can be proved that $|Z_i - Z_{i-1}| \leq 1$! why? prove it!

Azuma's inequality $\Rightarrow \mathcal{P}\left(|Z_n - \mathcal{E}Z_n| \geq t\right) \leq 2e^{-\frac{t^2}{2(n-1)}}.$

Remark: $\mathcal{P}\left(|\chi(G_{n,p}) - c| \geq t\sqrt{n-1}\right) \leq 2e^{-\frac{t^2}{2}}$ so the chromatic number is almost always concentrated on about $\sqrt{n}$ values.

QED