# Concentration Inequalities and Graph Coloring

Lecturer: Dr. Hong-Gwa Yeh
Department of Mathematics
National Central University
hgyeh@math.ncu.edu.tw

# A Simple Start

<u>Thm</u> (Markov's Inequality)

For $t > 0$, we have

$$P(|X| \geq t) \leq \frac{E|X|}{t}$$

pf:

$$\mathcal{E}|X| = \int_\Omega |X|\,d\mathcal{P} \geq \int_{|X|\geq t} |X|\,d\mathcal{P}$$

$$\geq t\,\mathcal{P}(|X|\geq t)$$

QED

# Chernoff's Inequality

<u>**Thm**</u>: $X_1, X_2, \cdots, X_n \overset{\text{i.i.d.}}{\sim} B(1,p)$.

Let $\lambda = np$ and $S_n = \sum_{i=1}^{n} X_i$.

Then, for $0 \le t \le n-\lambda$,

$$P\left\{ S_n \ge ES_n + t \right\} \le \left(\frac{\lambda}{\lambda+t}\right)^{\lambda+t} \left(\frac{n-\lambda}{n-\lambda-t}\right)^{n-\lambda-t}.$$

pf: For $u \geq 0$

$$\Pr\{S_n \geq ES_n + t\} \leq e^{-u(\lambda+t)} \prod_{i=1}^{n} E e^{uX_i}$$

$$= e^{-u(\lambda+t)} (1-p+pe^{u})^{n} \quad \bigstar$$

let $f(x) = x^{-(\lambda+t)}(1-p+px)^{n}$

$$f'(x) = -(\lambda+t)\, x^{-(\lambda+t+1)}(1-p+px)^{n} + np\, x^{-(\lambda+t)}(1-p+px)^{n-1}$$

and $f'(x)=0 \Rightarrow \hat{x} = \dfrac{(1-p)(\lambda+t)}{(n-\lambda-t)p}$

$f(x)$ attains its minimum at $\hat{x}$. assume $n-\lambda-t>0$

i.e. $\bigstar$ attains its minimum at $e^{u} = \hat{x}$ "

This yields

$$P\{S_n \geq ES_n + t\} \leq \left(\frac{(\lambda+t)(1-p)}{(n-\lambda-t)p}\right)^{-(\lambda+t)} \left(1-p+\frac{(1-p)(\lambda+t)}{(n-\lambda-t)}\right)^n$$

$$= \left(\frac{\lambda+t}{n-\lambda-t}\right)^{-(\lambda+t)} \left(\frac{1-p}{p}\right)^{-(\lambda+t)} (1-p)^n \left(\frac{n}{n-\lambda-t}\right)^n$$

Next we use the fact i.e. $p=\frac{\lambda}{n}$ to get

$$\Pr\{S_n \geq ES_n + t\} \leq \left(\frac{\lambda}{\lambda+t}\right)^{\lambda+t} \left(\frac{n-\lambda}{n-\lambda-t}\right)^{n-\lambda-t}$$

as $0 \leq t \leq n-\lambda$.

QED

# Chernoff's bounding technique

If $s$ is an arbitrary positive number then for any r.v. $X$ and any $t > 0$,

$$P(X \geq t) = P(e^{sX} \geq e^{st}) \leq \frac{\mathcal{E} e^{sX}}{e^{st}}.$$

In Chernoff's method, we find $s > 0$ that makes the upper bound small.

# Extensions of Chernoff's Ineq. can be derived from....

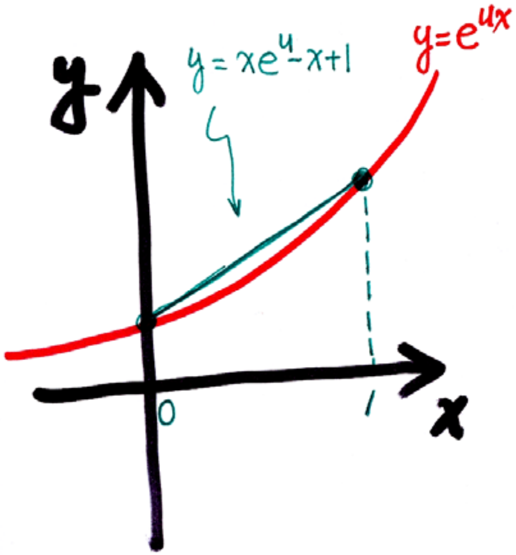
**Lemma***: Let $X_1, \cdots, X_n$ be independent with $0 \le X_i \le 1$ for each $i$. Let $p = \frac{\mathcal{E} S_n}{n}$, where $S_n = \sum_{i=1}^{n} X_i$.

Then for any $0 \le t < 1-p$,

$$P(S_n - \mathcal{E} S_n \ge nt) \le \left( \left(\frac{p}{p+t}\right)^{p+t} \left(\frac{1-p}{1-p-t}\right)^{1-p-t} \right)^n$$

**pf:** $P(S_n \ge np+nt) \le e^{-u(np+nt)} \prod_{i=1}^{n} \mathcal{E} e^{uX_i}$, where $u > 0$

$$\le e^{-u(np+nt)} \prod_{i=1}^{n} \mathcal{E}\left(1 - X_i + X_i e^{u}\right)$$




$$\le e^{-u(np+nt)} \left(\frac{n - \mathcal{E}S_n + (\mathcal{E}S_n)e^{u}}{n}\right)^{n}$$

geometric mean $\le$ arithmetic mean

$$= \left[e^{-u(p+t)}\left(1-p+pe^{u}\right)\right]^{n}$$

$=$ **RHS** by letting

$$e^{u} = \frac{(p+t)(1-p)}{p(1-p-t)}$$

**QED**

# Weaker but more useful bounds

**Thm:** Let $X_1, X_2, \cdots, X_n$ be independent rvs s.t. $0 \le X_i \le 1$ for each $i$. Let $p = \frac{ES_n}{n}$, where $S_n = \sum_{i=1}^n X_i$. Then

(a) For any $\epsilon > 0$,

$$P(S_n \ge (1+\epsilon)np) \le \exp\left(-np\left((1+\epsilon)\ln(1+\epsilon) - \epsilon\right)\right)$$
$$\le \exp\left(-\frac{\epsilon^2 np}{2(1+\epsilon/3)}\right)$$

(b) For any $\epsilon > 0$,

$$P(S_n \le (1-\epsilon)np) \le \exp\left(-\frac{1}{2}\epsilon^2 np\right)$$

**Pf of (a)** In the proof of Lemma*,

let $t = \epsilon p$ and $e^u = (1+\epsilon)$, then we have

$$P(S_n \geq (1+\epsilon)np)$$

$$= P(S_n \geq np + nt)$$

$$\leq \left[ (1+\epsilon)^{-(1+\epsilon)} (1+\epsilon p) \right]^n \quad \text{(by the proof of Lemma*)}$$

$$\leq \left[ (1+\epsilon)^{-(1+\epsilon)} (1+\epsilon p)^{\frac{1}{p}} \right]^{np}$$

$$\leq \left[ (1+\epsilon)^{-(1+\epsilon)} e^{\epsilon} \right]^{np} \quad (\because 1+\epsilon p \leq e^{\epsilon p})$$

this proves the first inequality in (a).

**pf of (a):** **Claim** For all $x \geq 0$, $(1+x)\ln(1+x) - x \geq \frac{3x^2}{6+2x}$.

Note that $* = e^{-np[(1+\epsilon)\ln(1+\epsilon) - \epsilon]}$

$$\leq e^{-np\frac{3\epsilon^2}{6+2\epsilon}}, \quad \text{done.}$$

**pf of (b):** Let $f(t) = \ln\left(\left(\frac{p}{p+t}\right)^{p+t}\left(\frac{1-p}{1-p-t}\right)^{1-p-t}\right)$

Let $h(x) = f(-xp)$ for $0 \leq x < 1$.

Then $h(0) = f(0) = 0$, $h'(0) = f'(0)(-p) = 0$ $\left(\because f'(t) = \ln\left(\frac{p(1-p-t)}{(p+t)(1-p)}\right)\right)$

$$h''(x) = f''(-xp)p^2 = -\frac{p}{(1-x)(1-p+xp)} \leq -p \quad (\because 0 \leq 1-x \leq 1,\ 0 \leq 1-p(1-x) \leq 1)$$

$\left(\text{Note } f''(t) = -\frac{1}{(p+t)(1-p-t)}\right)$

Taylor's Thm $\Rightarrow$ $h(x) = h(0) + h'(0)x + \frac{h''(\theta)x^2}{2}$, $0 \leq \theta \leq x$

$$\leq 0 + 0 - \frac{px^2}{2}$$

**Pf of (b)** Lemma* implies that

$$P\big((n-S_n)-(n-ES_n)\geq nt\big)\leq e^{f(t)n}$$

Let $t=\varepsilon p$. We have

$$P\big(S_n\leq(1-\varepsilon)ES_n\big)=P\big(S_n\leq ES_n-nt\big)\leq e^{f(t)n}$$

$$=e^{h(-\frac{t}{p})n}\leq e^{-\frac{p}{2}(\frac{t}{p})^2 n}=e^{-\frac{1}{2}\epsilon^2 np}.$$

$\left(\because\ h(x)\leq -\frac{px^2}{2},\ \text{for } 0\leq x<1\right)$

**QED**

**Remark:** The first inequality in (a) implies

$$P(S_n \geq 2ES_n) \leq e^{-(0.386)ES_n}$$ and

$$P(S_n \geq \delta ES_n) \leq e^{-\delta \ln(\frac{\delta}{e}) ES_n}$$

The second inequality in (a) implies

$$P(S_n \geq (1+\epsilon)ES_n) \leq e^{-\frac{1}{3}\epsilon^2 ES_n}$$

# Hoeffding's Lemma

**Lemma**: $\mathcal{E}X=0$, $a \le X \le b$.

Then for any $s>0$,

$$\mathcal{E}(e^{sX}) \le e^{\frac{s^2(b-a)^2}{8}}$$

**pf:** Note that $e^{sx} \le \left(\frac{x-a}{b-a}\right)(e^{sb}-e^{sa})+e^{sa}$ for $a \le x \le b$. It follows that


$$\mathcal{E} e^{sX} \le \frac{-a}{b-a}(e^{sb}-e^{sa})+e^{sa}$$

$$= e^{\phi(u)}, \text{ where } u \stackrel{def}{=} s(b-a),\ \phi(u) \stackrel{def}{=} -pu+\ln(1-p+pe^{u})$$

and $p \stackrel{def}{=} -\frac{a}{b-a}$

Note that $\phi'(u) = -p + \frac{p}{p+(1-p)e^{-u}}$, $\phi''(u) = \frac{p(1-p)e^{-u}}{(p+(1-p)e^{-u})^2} \le \frac{p(1-p)\frac{p}{1-p}}{\left(p+(1-p)\frac{p}{1-p}\right)^2} = \frac{1}{4}$

**Taylor's Thm** $\Rightarrow$ for some $\theta \in [0, u]$,

$$\phi(u) = \underbrace{\phi(0)}_{=0} + \underbrace{\phi'(0)}_{=0}u + \frac{\phi''(\theta)u^2}{2} \le \frac{u^2}{8} = \frac{s^2(b-a)^2}{8}$$

**QED**

# Hoeffding's Inequality

**Thm** Let $X_1, X_2, \cdots, X_n$ be independent rvs s.t. $a_i \le X_i \le b_i$ with probability one, $1 \le i \le n$.

Then for any $t \ge 0$ we have

$$P(S_n - ES_n \ge t) \le \exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$$

$$P(S_n - ES_n \le -t) \le \exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$$

Where $S_n = \sum_{i=1}^n X_i$.

证: $P(S_n - ES_n \geq t) = P\left(e^{s(S_n - ES_n)} \geq e^{st}\right)$

$$\leq e^{-st} E e^{s(S_n - ES_n)}$$

$$= e^{-st} \prod_{i=1}^{n} E e^{s(X_i - EX_i)}$$

$$\leq e^{-st} \prod_{i=1}^{n} e^{\frac{s^2 (b_i - a_i)^2}{8}}$$ by Hoeffding's Lemma

$(\because (b_i - EX_i) - (a_i - EX_i) = b_i - a_i)$

$$= e^{-st} e^{\frac{s^2}{8} \sum_{i=1}^{n} (b_i - a_i)^2}$$

$$= e^{-\frac{2t^2}{\sum_{i=1}^{n} (b_i - a_i)^2}} \quad \left(\text{by choosing } s = \frac{4t}{\sum_{i=1}^{n} (b_i - a_i)^2} > 0\right)$$

# Chebyshev's Ineq. is Inadequate

Let $X_1, X_2, \cdots \overset{iid}{\sim} B(1,p)$

Chebyshev's Inequality says that $P(|S_n - ES_n| \geq n\varepsilon) \leq \frac{p(1-p)}{n\varepsilon^2}$

However, as $n \to \infty$, Central Limit Thm suggests that

$$P(S_n - ES_n \geq n\varepsilon) = P\left(\frac{S_n - ES_n}{\sqrt{Var S_n}} \geq \frac{n\varepsilon}{\sqrt{np(1-p)}}\right) \sim 1 - \Phi\left(\frac{n\varepsilon}{\sqrt{np(1-p)}}\right)$$

$$\leq \exp\left(-\frac{n\varepsilon^2}{2p(1-p)}\right) \quad \left(\because \sqrt{2\pi}\{1-\Phi(x)\} < \frac{1}{x}\exp\left(-\frac{x^2}{2}\right)\right)$$

It seems Hoeffding Ineq. is adequate .....

# However....

Hoeffdingó bound is independent of p, so it may loose if p is small (or large).

That is this inequality ignores information about the variance of the $X_i$´s.

The **Bernstein's Inequality** will give an answer.....

# A Lemma first

<u>**Lemma:**</u> If $|X| \le c$, $\mathcal{E}X = 0$ and $Var(X) = \sigma^2$,

then $\mathcal{E}\left(e^{sX}\right) \le \exp\left(\frac{\sigma^2}{c^2}\left(e^{sc} - 1 - sc\right)\right)$

Pf:

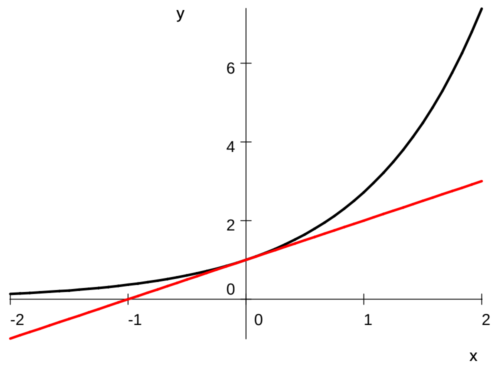
$$e^{sx} = 1 + sx + \sum_{r=2}^{\infty} \frac{(sx)^r}{r!}$$

$$\Rightarrow \quad E\, e^{sX} = 1 + \sum_{r=2}^{\infty} \frac{s^r E X^r}{r!} \qquad (\because EX = 0)$$

$$\le 1 + \sum_{r=2}^{\infty} \frac{s^r c^{r-2} \sigma^2}{r!} \qquad \left(\because EX^r \le E|X|^{r-2}X^2 \le c^{r-2}\sigma^2 \text{ for } r \ge 2\right)$$

$$= 1 + \frac{\sigma^2}{c^2} \sum_{r=2}^{\infty} \frac{(sc)^r}{r!}$$

$$= 1 + \frac{\sigma^2}{c^2}\left(e^{sc} - 1 - sc\right)$$




$$\le e^{\frac{\sigma^2}{c^2}(e^{sc} - 1 - sc)} \qquad (\because e^x \ge 1 + x \quad \forall x)$$

# The Bennett's Inequality

**Thm** Let $X_1, X_2, \cdots, X_n$ be independent rvs with $EX_i = 0$ and $|X_i| \le c$, $1 \le i \le n$. Let $S_n = \sum_{i=1}^{n} X_i$ and $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} Var(X_i)$. Then for any $t > 0$,

$$P(S_n > t) \le \exp\left(-\frac{t}{c}\left(\left(1+\frac{n\sigma^2}{ct}\right)\ln\left(1+\frac{ct}{n\sigma^2}\right)-1\right)\right)$$

# The Proof

**Proof:**

$$P(S_n > t) \leq e^{-st} E e^{sS_n} \quad \text{(Markov's ineq.)}$$

$$= e^{-st} \prod_{i=1}^{n} E e^{sX_i}$$

$$\leq e^{-st} \prod_{i=1}^{n} e^{\frac{\sigma_i^2}{c^2}(e^{sc} - 1 - sc)} \quad \text{by above lemma}$$

$$= e^{-st} e^{\frac{n\sigma^2}{c^2}(e^{sc} - 1 - sc)}$$

The last term is minimized for

$$s = \frac{1}{c} \ln\left(1 + \frac{tc}{n\sigma^2}\right).$$

Resubstituting this value, we obtain Bennett's ineq.

# Bernstein's Inequality

**Thm** Under the conditions of Bennett's Ineq., for $\varepsilon > 0$, we have

$$\mathbb{P}(S_n > n\varepsilon) \leq \exp\left(-\frac{n\varepsilon^2}{2\sigma^2 + c\varepsilon}\right)$$

pf: Applying the following elementary ineq. to Bennett's bound: $\ln(1+x) \geq \frac{2x}{2+x}$, $x \geq 0$

ln(1+x)= x-(1/2)x²+(1/3)x³+O(x⁴)
(2x)/(2+x)= x-(1/2)x²+(1/4)x³+O(x⁴)

Bennett's bound

$$\leq \exp\left(-\frac{n\varepsilon}{c}\left(\left(1+\frac{n\sigma^2}{c\varepsilon n}\right)\left(\frac{2\frac{cn\varepsilon}{n\sigma^2}}{2+\frac{cn\varepsilon}{n\sigma^2}}\right)-1\right)\right)$$

$$= \exp\left(-\frac{n\varepsilon}{c}\left(\frac{c\varepsilon+\sigma^2}{c\varepsilon}\,\frac{2c\varepsilon}{c\varepsilon+2\sigma^2}-1\right)\right)$$

$$= \exp\left(-\frac{n\varepsilon}{c}\,\frac{c\varepsilon}{2\sigma^2+c\varepsilon}\right) = \exp\left(-\frac{n\varepsilon^2}{2\sigma^2+c\varepsilon}\right)$$

QED

# What's after Bernstein's ineq.....

## Poisson-type Inequality

<u>Thm</u> Let $X_1, X_2, \cdots, X_n$ be independent r.v.s with $0 \le X_i \le 1$ $\forall i$ and $m = \mathcal{E} S_n$.

Then for any $t \ge m$,

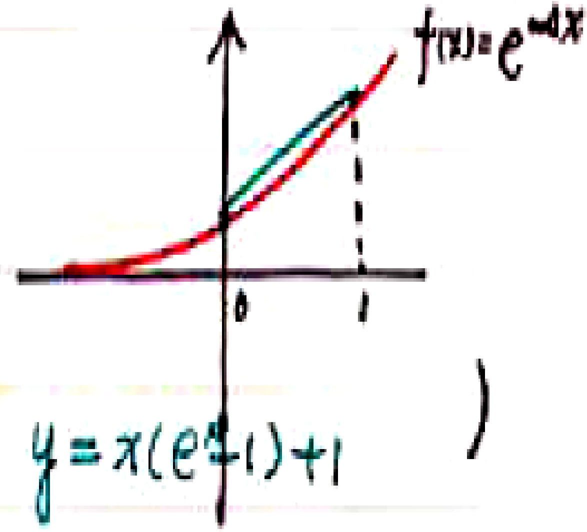
$$P(S_n \ge t) \le \left(\frac{m}{t}\right)^t e^{t-m}$$

<u>Note</u>: Here $n$ doesn't appear on the RHS

pf: let $m_i = EX_i$.

$$P(S_n \ge t) \le e^{-\lambda t}\prod_{i=1}^{n} E e^{\lambda X_i}$$

$$\le e^{-\lambda t}\prod_{i=1}^{n} E\big(X(e^{\lambda}-1)+1\big) \quad (\because$$




$)$

$$= e^{-\lambda t}\prod_{i=1}^{n}\left[1+m_i(e^{\lambda}-1)\right]$$

$$\le e^{-\lambda t}\prod_{i=1}^{n} e^{m_i(e^{\lambda}-1)} = e^{-\lambda t}e^{m(e^{\lambda}-1)} \quad ✱$$

$$\le \left(\frac{m}{t}\right)^{t} e^{t-m}$$ by choosing $\lambda = \ln\left(\frac{t}{m}\right)$

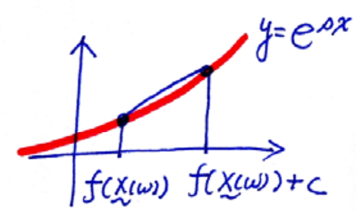
to minimizes the bound ✱ QED

# An extension of Hoeffding's Lemma

**Lemma:** Suppose for rv $V$ and r.vector $\underset{\sim}{X}$ we have

① $\mathcal{E}(V \mid \underset{\sim}{X}) = 0$ a.s. ② $f(\underset{\sim}{X}) \le V \le f(\underset{\sim}{X}) + c$, for some fun. $f$ and constant $c$.

Then for any $s > 0$, $\mathcal{E}(e^{sV} \mid \underset{\sim}{X}) \le e^{\frac{s^2c^2}{8}}$.




**pf:** Note that

$$e^{sV} \le \frac{V - f(\underset{\sim}{X})}{c}\left(e^{s(f(\underset{\sim}{X})+c)} - e^{sf(\underset{\sim}{X})}\right) + e^{sf(\underset{\sim}{X})}$$

Thus $\mathcal{E}(e^{sV} \mid \underset{\sim}{X}) = \frac{-f(\underset{\sim}{X})}{c}\left(e^{s(f(\underset{\sim}{X})+c)} - e^{sf(\underset{\sim}{X})}\right) + e^{sf(\underset{\sim}{X})}$.

The remains is similar to Hoeffding's Lemma.

# McDiarmid's Inequality

**Thm:** Let $X_1, X_2, \cdots, X_n$ be independent rvs. Let $f$ be a function $f: \mathbb{R}^n \to \mathbb{R}$ with a vector $(c_1, c_2, \cdots, c_n)$ s.t. $|f(\underset{\sim}{x}) - f(\underset{\sim}{y})| \le c_i$ for all $\underset{\sim}{x}, \underset{\sim}{y}$ in $\mathbb{R}^n$ that differ only at the $i$th coordinate, $1 \le i \le n$.

Then for any $t > 0$,

$$P\left( \left| f(X_1, \cdots, X_n) - \mathcal{E} f(X_1, \cdots, X_n) \right| \ge t \right) \le 2 e^{-\frac{2t^2}{\sum_{i=1}^n c_i^2}}$$

**pf:** Let $\underset{\sim}{X} = (X_1, \cdots, X_n)$, $Z_0 = \mathcal{E} f(\underset{\sim}{X})$, $Z_i = \mathcal{E}(f(\underset{\sim}{X}) \mid X_1, \cdots, X_i)$, $Z_n = f(\underset{\sim}{X})$.

**Claim** $\mathcal{E}\left(e^{s(Z_k - Z_{k-1})} \mid X_1, \cdots, X_{k-1}\right) \le e^{\frac{s^2 c_k^2}{8}}$, $1 \le k \le n$.

**pf:** Let $U_k = \sup_u \left\{ \mathcal{E}\left(f(\underset{\sim}{X}) \mid X_1, \cdots, X_{k-1}, u\right) - \mathcal{E}\left(f(\underset{\sim}{X}) \mid X_1, \cdots, X_{k-1}\right) \right\}$

$L_k = \inf_l \left\{ \mathcal{E}\left(f(\underset{\sim}{X}) \mid X_1, \cdots, X_{k-1}, l\right) - \mathcal{E}\left(f(\underset{\sim}{X}) \mid X_1, \cdots, X_{k-1}\right) \right\}$

Note that $U_k - L_k \le \sup_{l,u} \left\{ \mathcal{E}\left(f(\underset{\sim}{X}) \mid X_1, \cdots, X_{k-1}, u\right) - \mathcal{E}\left(f(\underset{\sim}{X}) \mid X_1, \cdots, X_{k-1}, l\right) \right\}$

$\because X_1, \cdots, X_n$ are independent

$\le \sup_{l,u} \left\{ \sum_{y_{k+1}, \cdots, y_n} \left[ f(X_1, \cdots, X_{k-1}, u, y_{k+1}, \cdots, y_n) - f(X_1, \cdots, X_{k-1}, l, y_{k+1}, \cdots, y_n) \right] \prod_{j=k+1}^{n} P(X_j = y_j) \right\}$

$\le c_k$ ($\because$ Lipschitz condition)

So $L_k \le Z_k - Z_{k-1} \le U_k \le L_k + c$ and hence the claim is true by the extension of Hoeffding's lemma, since $\mathcal{E}(Z_k - Z_{k-1} \mid X_1, \cdots, X_{k-1}) = 0$ a.s.

QED of claim

**Pf (continued)**

$$\Pr\{ f(\underset{\sim}{X}) - \mathcal{E} f(\underset{\sim}{X}) \geq t \}$$

$$\leq e^{-st}\, \mathcal{E}\, e^{s[f(\underset{\sim}{X}) - \mathcal{E} f(\underset{\sim}{X})]}$$

$$= e^{-st}\, \mathcal{E}\, e^{s \sum_{k=1}^{n} [Z_k - Z_{k-1}]}$$

$$= e^{-st}\, \mathcal{E}\left(\mathcal{E}\left(e^{s \sum_{k=1}^{n} [Z_k - Z_{k-1}]} \,\middle|\, X_1, \cdots, X_{n-1}\right)\right) \quad (\because \text{Tower property})$$

$$= e^{-st}\, \mathcal{E}\left(e^{s \sum_{k=1}^{n-1} [Z_k - Z_{k-1}]}\, \mathcal{E}\left(e^{s(Z_n - Z_{n-1})} \,\middle|\, X_1, \cdots, X_{n-1}\right)\right)$$

$$\leq e^{-st}\, \mathcal{E}\left(e^{s \sum_{k=1}^{n-1} [Z_k - Z_{k-1}]}\, e^{\frac{s^2 c_n^2}{8}}\right) \quad (\text{by the claim})$$

$$\leq e^{-st} \prod_{k=1}^{n} e^{\frac{s^2 c_k^2}{8}} \quad (\text{by repeating the same argument } n \text{ times})$$

$$= e^{-st + s^2 \sum_{k=1}^{n} \frac{c_k^2}{8}} \leq e^{-\frac{2t^2}{\sum_{k=1}^{n} c_k^2}} \quad \text{by choosing } s = \frac{4t}{\sum_{k=1}^{n} c_k^2}$$

**QED**

# Bin Packing

- The Bin Packing Problem requires finding the minimum number of unit size bins needed to pack a given collection of items with sizes in [0,1]. In our model, we have n items to pack, and the size of items are $X_1,\ldots,X_n$, i.i.d. over [0,1]. *We use a fixed procedure for packing*.

Ref:
Rhee & Talagrand (1987) "Martingale inequalities and NP-complete problem" Math. of Oper. Res., 12, 177-181.

# Bin Packing

**Discussion:** We denote by $f(X_1, \cdots, X_n)$ the numbers of bins needed to pack $X_1, X_2, \cdots, X_n$ using procedure $\mathcal{A}$.

(a) If $\mathcal{A}$ is optimum then $f$ is 1-Lipschitz. McDiarmid's Ineq. says that $P(|f(\underset{\sim}{X}) - \mathcal{E}f(\underset{\sim}{X})| \geq t) \leq 2e^{-\frac{2t^2}{n}}$.

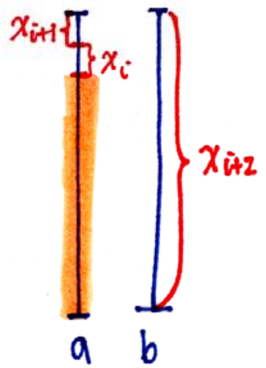
(b) If $\mathcal{A}$ is the procedure Next Fit then $f$ is 2-Lipschitz, and hence $P(|f(\underset{\sim}{X}) - \mathcal{E}f(\underset{\sim}{X})| \geq t) \leq 2e^{-\frac{t^2}{n}}$.

# Next Fit Procedure

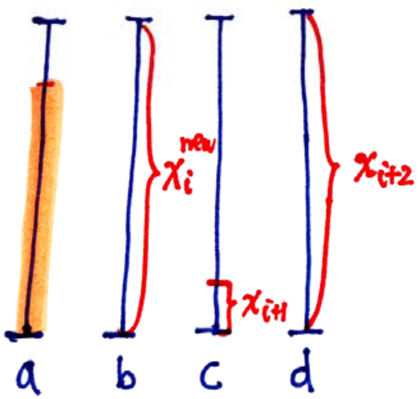
- Where the bins are filled one at a time and a new bin is started when the current element does not fit in the bin being currently filled.

To see $f$ is 2-Lipschitz:

$(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$




$(x_1, \cdots, x_{i-1}, x_i^{new}, x_{i+1}, \cdots, x_n)$



# Concentration of the Chromatic Number

**Thm** (Shamir-Spencer) Let $n \geq 2$ and $p \in (0,1)$.

Then we have

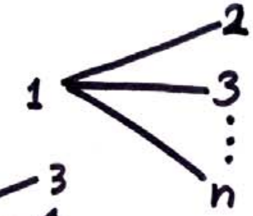
$$P\left(\left|\chi(G_{n,p}) - \mathcal{E}\chi(G_{n,p})\right| \geq t\right) \leq 2e^{-\frac{2t^2}{n-1}}$$

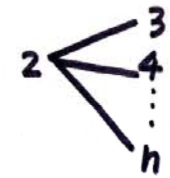
**Remark**: probability space $\Omega = \{\omega : \omega \text{ is a graph on the vertex set } \{1,2,\cdots,n\}\}$

with probability measure $P(\omega) = p^m (1-p)^{\binom{n}{2}-m}$ where $m = |E(\omega)|$.

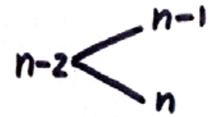
One may consider $G_{n,p}$ as a random element $G_{n,p} : \Omega \to \Omega$ s.t. $G_{n,p}(\omega) = \omega$.

Sometimes we consider $G_{n,p}$ as a prob. space.

**pf:** Let $\underset{\sim}{X_1} = (X_{12}, X_{13}, X_{14}, \cdots, X_{1n})$ 

$\underset{\sim}{X_2} = (X_{23}, X_{24}, \cdots\cdots, X_{2n})$ 

$\vdots$

$\underset{\sim}{X_{n-2}} = (X_{n-2,n-1}, X_{n-2,n})$ 

$\underset{\sim}{X_{n-1}} = X_{n-1,n}$ 

We have $\chi(G_{n,p}) = f(\underset{\sim}{X_1}, \underset{\sim}{X_2}, \cdots, \underset{\sim}{X_{n-1}})$ for some function $f$.

Note that $f: S_1 \times S_2 \times \cdots \times S_{n-1} \longrightarrow \mathbb{R}$, where $S_i = \{0,1\}^{n-i}$, and

$f$ is a 1-Lipschitz function.

By McDiarmid's Inequality, we arrive at

$$P\left(|\chi(G_{n,p}) - E\chi(G_{n,p})| \geq t\right) \leq 2e^{-\frac{2t^2}{n-1}}$$

QED

# A Remark on Shamir-Spencer's Thm.

- Since $\mathcal{P}(|\chi(G_{n,p})-\mathcal{E}\chi(G_{n,p})|\geq t\sqrt{n-1})\leq 2e^{-2t^2}$, so the chromatic number is almost always concentrated on about $O(\sqrt{n})$ values.
- **Note that we have no clue to what the value of $\mathcal{E}\chi(G_{n,p})$ is.**

# A technique lemma...

**Thm** Lemma 7.3.4 (Alon & Spencer) Let $\alpha, c$ be fixed and $\alpha > \frac{5}{6}$. let $p = n^{-\alpha}$. Then almost always every $c\sqrt{n\log n}$ vertices of $G_{n,p}$ induce a 3-colorable subgraph.

**pf**: $\mathcal{P}(\forall \text{ subgraph } H \text{ of } G_{n,p} \text{ with } \nu_H \le c\sqrt{n\log n} \text{ having } \chi(H) \le 3)$

$$= 1 - \mathcal{P}\left(\bigcup_{t=4}^{c\sqrt{n\log n}} \{\exists H \subseteq G_{n,p} \text{ with } \nu_H = t \text{ having } \chi(H) > 3\}\right)$$

$$\star \le \sum_{t=4}^{c\sqrt{n\log n}} \mathcal{P}(\exists \text{ minimal } H \subseteq G_{n,p} \text{ with } \nu_H = t \text{ having } \chi(H) > 3)$$

i.e. $\chi(H - v) \le 3$ for $\forall v \in V(H)$

$$\le \sum_{t=4}^{c\sqrt{n\log n}} \mathcal{P}(\exists H \subseteq G_{n,p} \text{ with } \nu_H = t \text{ having } d_H(x) \ge 3 \text{ for all } x \in V(H))$$

$$\le \sum_{t=4}^{c\sqrt{n\log n}} \binom{n}{t} \mathcal{P}(H \cong G_{n,p}[\{1,2,\cdots,t\}] \text{ has } e_H \ge \tfrac{3}{2}t)$$

Pf (Continued)

$$\leq \sum_{t=4}^{c\sqrt{n\log n}} \binom{n}{t} \binom{\binom{t}{2}}{\frac{3}{2}t} p^{\frac{3}{2}t}$$

$$\leq \sum_{t=4}^{c\sqrt{n\log n}} \left(\frac{en}{t}\right)^t \left(\frac{e\binom{t}{2}}{\frac{3}{2}t}\right)^{\frac{3}{2}t} p^{\frac{3}{2}t} \quad \left(\because \binom{n}{k} \leq \left(\frac{en}{k}\right)^k\right)$$

$$\leq \sum_{t=4}^{c\sqrt{n\log n}} \left(\frac{en}{t} \frac{t^{\frac{3}{2}} e^{\frac{3}{2}}}{3^{\frac{3}{2}}} n^{-\frac{3}{2}\alpha}\right)^t \quad (\because p = n^{-\alpha})$$

$$\leq \sum_{t=4}^{c\sqrt{n\log n}} \left(c_1 n^{1-\frac{3}{2}\alpha} t^{\frac{1}{2}}\right)^t$$

$$\leq \sum_{t=4}^{c\sqrt{n\log n}} \left(c_2 n^{1-\frac{3}{2}\alpha} n^{\frac{1}{4}} (\log n)^{\frac{1}{4}}\right)^t$$

$$= \sum_{t=4}^{c\sqrt{n\log n}} \left(c_2 n^{-\varepsilon} (\log n)^{\frac{1}{4}}\right)^t \quad \left(\because \frac{5}{4} - \frac{3}{2}\alpha < 0 \Leftrightarrow \frac{5}{6} < \alpha\right)$$

$$\leq \left(c_2 n^{-\varepsilon} (\log n)^{\frac{1}{4}}\right)^4 \frac{1}{1 - c_2 n^{-\epsilon} (\log n)^{\frac{1}{4}}} = o(1)$$

QED

# Four-Value Concentration

**Thm** (see Alon & Spencer Thm 7.3.3) Let $\alpha > \frac{5}{6}$ be fixed, and let $p = n^{-\alpha}$ (i.e. $p$ is not too large)

Then for any $n$, $\exists\ u = u(\alpha, n)$ such that

$$\chi(G_{n,p}) \in \{u, u+1, u+2, u+3\} \text{ almost surely}$$

i.e. $\Pr\{\chi(G_{n,p}) \notin \{u, u+1, u+2, u+3\}\} \to 0$ as $n \to \infty$

**pf:** Let $u = u(n,\alpha)$ be the smallest integer s.t. $P(\chi(G_{n,p}) \le u) > \frac{1}{n}$.

Claim A: $P(\chi(G_{n,p}) \ge u) \ge 1 - \frac{1}{n}$

pf: The choice of $u \Rightarrow P(\chi(G_{n,p}) \le u-1) \le \frac{1}{n} \Rightarrow P(\chi(G_{n,p}) \ge u) \ge 1 - \frac{1}{n}$ ∎A

Let $X$ be the minimum number of vertices whose deletion makes $G_{n,p}$ $u$-colorable. Then $X = f(\underset{\sim}{X_1}, \underset{\sim}{X_2}, \cdots, \underset{\sim}{X_{n-1}})$ for some function $f$, where $\underset{\sim}{X_1}, \underset{\sim}{X_2}, \cdots, \underset{\sim}{X_{n-1}}$ were defined in the proof of Shamir-Spencer's Thm. Note that $f$ is 1-Lipschitz.

Claim B: $\sqrt{2(n-1)\log n} > \mathcal{E}X$

pf:
$$
\begin{aligned}
\frac{1}{n} < P(\chi(G_{n,p}) \le u) &= P(X=0) \\
&= P(X \le \mathcal{E}X - \mathcal{E}X) \\
&= P(f(\underset{\sim}{X_1}, \cdots, \underset{\sim}{X_{n-1}}) \le \mathcal{E}f(\underset{\sim}{X_1}, \cdots, \underset{\sim}{X_{n-1}}) - \mathcal{E}X) \\
&\le \exp\left(-\frac{2(\mathcal{E}X)^2}{n-1}\right) \quad \text{by McDiarmid's Inequality}
\end{aligned}
$$

Therefore $\sqrt{\frac{1}{2}(n-1)\log n} > \mathcal{E}X$. ∎B

## pf (Continued)

Claim C: $P\left(X < 2\sqrt{2(n-1)\log n}\right) \geq 1 - \frac{1}{n}$

pf: $\text{LHS} = 1 - P\left(X \geq 2\sqrt{2(n-1)\log n}\right)$

$$\geq 1 - P\left(X \geq EX + \sqrt{2(n-1)\log n}\right)$$

$$\geq 1 - \exp\left(-\frac{2(n-1)\log n}{2(n-1)}\right) \quad \text{by McDiarmid's Ineq. again!}$$

$$= 1 - \frac{1}{n}. \qquad \blacksquare_C$$

Let $A = \{\chi(G_{n,p}) \geq u\}$ and $B = \{X < 2\sqrt{2(n-1)\log n}\}$ be two events.

Then $P(A \cap B) = 1 - P(\bar{A} \cup \bar{B}) \geq 1 - P(\bar{A}) - P(\bar{B})$

$$\geq 1 - \frac{1}{n} - \frac{1}{n} = 1 - \frac{2}{n} \longrightarrow 1 \text{ as } n \to \infty$$

The thm. follows from the technique lemma we proved.

**QED**

# Kim's Lemma

**Thm** (J.H. Kim 1995) Suppose that

- $X_1, X_2, \cdots, X_n$ are independent rvs s.t. $X_i \sim B(1, p_i)$ $\forall i$.
- $f: \{0,1\}^n \longrightarrow \mathbb{R}$ and $g$ is a convex function.

Then for any $i$, $1 \le i \le n$, we have

$q_i = 1 - p_i$

$$\mathcal{E}\Big(g(V_i) \,\Big|\, X_1, \cdots, X_{i-1}\Big) \le \mathcal{E}\Big(p_i\, g(q_i \gamma_i) + q_i\, g(-p_i \gamma_i) \,\Big|\, X_1, \cdots, X_{i-1}\Big)$$

where $\gamma_i = f(X_1, \cdots, X_{i-1}, 1, X_{i+1}, \cdots, X_n) - f(X_1, \cdots, X_{i-1}, 0, X_{i+1}, \cdots, X_n)$

$$V_i = \mathcal{E}\big(f(\underset{\sim}{X}) \,|\, X_1, \cdots, X_i\big) - \mathcal{E}\big(f(\underset{\sim}{X}) \,|\, X_1, \cdots, X_{i-1}\big)$$

$\underset{\sim}{X} = (X_1, \cdots, X_n)$

# An extension of Devroye's Inequality

**Thm** If we have

(1) $X_1, X_2, \cdots, X_n$ are independent, $X_i \sim B(1, p_i)$ $\forall i$

(2) $f : \{0,1\}^n \longrightarrow \mathbb{R}$ having a vector $(c_1, c_2, \cdots, c_n)$ s.t.

$|f(\underset{\sim}{a}) - f(\underset{\sim}{b})| \leq c_i$ for all $\underset{\sim}{a}, \underset{\sim}{b}$ that differ only in the ith coordinate.

Then

$$\mathrm{Var}[f(X_1, X_2, \cdots, X_n)] \leq \sum_{i=1}^{n} c_i^2 p_i (1-p_i)$$

**pf**: Let $f = f(X_1, X_2, \cdots, X_n)$ and $V_i = \mathcal{E}(f \mid \underbrace{X_1, \cdots, X_i}_{\underset{\sim}{X_i}}) - \mathcal{E}(f \mid \underbrace{X_1, \cdots, X_{i-1}}_{\underset{\sim}{X_{i-1}}})$ $2 \le i \le n$

$V_1 \overset{\text{def}}{=} \mathcal{E}(f \mid X_1) - \mathcal{E}(f \mid \mathcal{F}_0)$

Then $\text{Var} f = \mathcal{E}(f - \mathcal{E}f)^2$

$$= \mathcal{E}\left( \left(\sum_{i=1}^n V_i\right)^2 \right)$$

$\because$ $\mathcal{E}(f \mid \mathcal{F}_0) = \mathcal{E}f$ where $\mathcal{F}_0 = \{\phi, \Omega\}$

$\mathcal{E}(f \mid \underset{\sim}{X_n}) = f$

$$= \mathcal{E}\left( \sum_{i=1}^n V_i^2 + 2 \sum_{1 \le i < j \le n} V_i V_j \right)$$

$$= \mathcal{E}\left( \sum_{i=1}^n V_i^2 \right)$$

$\because$ $\mathcal{E} V_i V_j = \mathcal{E}\left( \mathcal{E}(V_i V_j \mid X_1, \cdots, X_{j-1}) \right)$ $1 \le i < j \le n$

$= \mathcal{E}\left( V_i \, \mathcal{E}(V_j \mid X_1, \cdots, X_{j-1}) \right) = \mathcal{E}(V_i \cdot 0) = 0$

$$= \sum_{i=1}^n \mathcal{E}\left( \mathcal{E}(V_i^2 \mid \underset{\sim}{X_{i-1}}) \right)$$

here we define $\mathcal{E}(V_1^2 \mid \underset{\sim}{X_0}) \overset{\text{def}}{=} \mathcal{E}(V_1^2 \mid \mathcal{F}_0)$

$$\le \sum_{i=1}^n \mathcal{E}\left( \mathcal{E}\left( p_i (q_i r_i)^2 + q_i (p_i r_i)^2 \mid \underset{\sim}{X_{i-1}} \right) \right)$$

Applying Kim's Lemma with $g(x) = x^2$.

$$= \sum_{i=1}^n p_i q_i r_i^2$$

$$\le \sum_{i=1}^n p_i q_i c_i^2$$

$\because$ $|r_i| \le c_i$ Lipschitz property

**QED**

- $k$-$L(2,1)$-labeling (coloring) of $G$:

$\varphi : V(G) \longrightarrow \{0, 1, 2, \cdots, k\}$ such that

$x \sim y \implies |\varphi(x) - \varphi(y)| \geq 2$

$x \wedge y \implies |\varphi(x) - \varphi(y)| \geq 1$

- $\lambda_{2,1}(G) \overset{\text{def}}{=} \min\{k : G \text{ has a } k\text{-}L(2,1)\text{-labeling (coloring)}\}$

coloring

- $k$-$L(a,b)$-labeling of $G$:

$\varphi : V(G) \longrightarrow \{0, 1, 2, \cdots, k\}$ such that

$x \sim y \implies |\varphi(x) - \varphi(y)| \geq a$


$\implies |\varphi(x) - \varphi(y)| \geq b$

- $\lambda_{a,b}(G) \stackrel{\text{def}}{=\!=} \min\{k : G \text{ has a } k\text{-}L(a,b)\text{-labeling}\}$

coloring

# Something New

Theorem Let $n \geq 2$ and $p \in (0,1)$ be arbitrary, and let $c = E[\lambda_{2,1}(G_{n,p})]$.
Then

$$\Pr\left\{ |\lambda_{2,1}(G_{n,p}) - c| \geq t\sqrt{n(n-1)} \right\} \leq 2e^{-t^2}$$

Pf:

- $\lambda_{2,1}(G_{n,p})$ can be expressed as a function of $X_{e_1}, \cdots, X_{e_m}$, $m = \binom{n}{2}$
- $\lambda_{2,1}(G_{n,p})$ is a 2-Lipschitz function in the $X_{e_i}$'s
- McDiarmid's inequality now gives what we want.

# Another Something New

**Thm** For any $n$, there is a $u = u(n)$ such that

$$\lambda_{2,1}(G_{n,p}) \in \left[u,\ u + 4\sqrt{n(n-1)\log n} + 1\right]$$

almost surely.

# Sparse Random Graphs

Observation: If $p \leq 1/n\log n$ and $c = E[\lambda_{2,1}(G_{n,p})]$ then

$$\Pr\{|\lambda_{2,1}(G_{n,p}) - c| \geq \sqrt{n}\}$$

$$\leq \frac{\mathrm{Var}\{\lambda_{2,1}(G_{n,p})\}}{n}$$

$$\leq \frac{\sum_{i=1}^{\binom{n}{2}} 4p(1-p)}{n}$$

$$\leq \frac{2}{\log n} \longrightarrow 0 \quad \text{as } n \to \infty$$

# Thank you