On $L(2,1)$-Labeling Problem

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Abstract

An $L^\lambda(2,1)$-labeling of a graph G is an assignment of colors from the integer set $\{0,1,2,\ldots,\lambda\}$ to the vertices of the graph G such that vertices at distance at most two get different colors and adjacent vertices get colors which are at least two apart. The minimum value λ for which G admits an $L^\lambda(2,1)$-labeling is denoted by $\lambda_{2,1}(G)$. In this note some recent results on $L(2,1)$-labeling problem are surveyed, and new results are also given for $\lambda_{2,1}$ and its variations by using probabilistic tools.

1 $L(2,1)$-coloring of random graphs

Theorem 1.1 *Let $n \geq 2$ and $p \in (0,1)$ be arbitrary, and let $c = c(n,p) = E[\lambda_{2,1}(G_{n,p})]$. Then*

$$\Pr\{|\lambda_{2,1}(G_{n,p}) - c| \geq t\sqrt{n(n-1)}\} \leq 2e^{-t^2}.$$

Proof. Let $e_1, e_2, \ldots, e_m$ be the edges of $G_{n,p}$ enumerated in a fixed order, let X_{e_i} be the indicator random variable showing the presence or absence of the edge e_i in $G_{n,p}$. The $L(2,1)$-labeling number $\lambda_{2,1}(G_{n,p})$, can be expressed as a function of $X_{e_1}, \ldots, X_{e_m}$ which are independent. Moreover, $\lambda_{2,1}(G_{n,p})$ is a 2-Lipschitz function in the X_{e_i}'s because changing one edge of a graph only changes the $L(2,1)$-number by at most 2. Indeed, suppose graph H is obtained from graph G by deleting one edge $e = xy$. Let h be an $L(2,1)$-labeling of H from V_H onto the set $\{0,1,2,\ldots,\lambda_{2,1}(H)\}$. Define a function g from V_G to the set $\{0,1,2,\ldots,\lambda_{2,1}(H)+2\}$ as follows: If $f(x) \neq f(y)$, say $f(x) < f(y)$, then define

$$g(v) = \begin{cases} f(v) & \text{if } v \neq x \text{ and } f(v) \leq f(x), \\ f(v)+1 & \text{if } (v = x) \text{ or } (v \neq y \text{ and } f(x) < f(v) \leq f(y)), \\ f(v)+2 & \text{if } v = y \text{ or } f(y) < f(v). \end{cases}$$

If $f(x) = f(y)$, then define

$$g(v) = \begin{cases} f(v) & \text{if } (v = x) \text{ or } (f(v) < f(x)), \\ f(v)+1 & \text{if } v \notin \{x,y\} \text{ and } f(v) = f(x), \\ f(v)+2 & \text{if } (v = y) \text{ or } (f(x) < f(v)). \end{cases}$$

A straightforward discussion shows that g is an $L(2,1)$-labeling of G from V_G to the set $\{0,1,2,\ldots,\lambda_{2,1}(H)+2\}$ and hence $\lambda_{2,1}(G) - 2 \leq \lambda_{2,1}(H) \leq \lambda_{2,1}(G)$.

McDiarmid's inequality now gives that

$$\Pr\{|\lambda_{2,1}(G_{n,p}) - c| \geq t\sqrt{n(n-1)}\} \leq 2e^{-t^2}$$

∎

Theorem 1.2 *For any n, there is a $u = u(n)$ such that*

$$\lambda_{2,1}(G_{n,p}) \in [u, u + 4\sqrt{n(n-1)\log n} + 1]$$

almost surely.

Proof. Let u be the smallest integer such that $\Pr(\lambda_{2,1}(G_{n,p}) \leq u) > 1/n$. Let X be the minimum number of edges whose deletion makes $G_{n,p}$ u-$L(2,1)$-colorable. X can be expressed as a function of independent random variables $X_{e_1}, \ldots, X_{e_m}$, where X_{e_i} indicates the presence or absence of the edge e_i in $G_{n,p}$. It is clear that X is a 2-Lipschitz function of $X_{e_1}, \ldots, X_{e_m}$. By McDiarmid's inequality we claim that $EX < \sqrt{n(n-1)\log n}$. Indeed, since

$$\begin{aligned} \frac{1}{n} &< \Pr(\lambda_{2,1}(G_{n,p}) \leq u) \\ &= \Pr(X = 0) \\ &\leq \Pr(X \leq EX - EX) \\ &\leq \exp\left(-\frac{2(EX)^2}{4\binom{n}{2}}\right) \end{aligned}$$

McDiarmid's inequality also leads to

$$\begin{aligned} \Pr(X < 2\sqrt{n(n-1)\log n}) &= 1 - \Pr(X \geq 2\sqrt{n(n-1)\log n}) \\ &\geq 1 - \Pr(X \geq EX + \sqrt{n(n-1)\log n}) \\ &\geq 1 - \exp\left(-\frac{2n(n-1)\log n}{4\binom{n}{2}}\right) \\ &= 1 - \frac{1}{n} \end{aligned}$$

We conclude that $\lambda_{2,1}(G_{n,p}) \in [u, u + 4\sqrt{n(n-1)\log n} + 1]$ almost surely. Indeed, since

$$\begin{aligned} \Pr(\lambda_{2,1}(G_{n,p}) \geq u,\ X < 2\sqrt{n(n-1)\log n}) &\geq \Pr(X < 2\sqrt{n(n-1)\log n}) - \Pr(\lambda_{2,1}(G_{n,p}) < u) \\ &\geq 1 - \frac{2}{n} \to 1 \end{aligned}$$

as $n \to \infty$. ∎