

# Bregman's Thm (I)

$$A = [a_{ij}] \in M_{n \times n} [0, 1]$$

$$\text{per}(A) \stackrel{\text{def}}{=} \sum_{\sigma \in S_n} a_{1\sigma_1} a_{2\sigma_2} \dots a_{n\sigma_n}$$

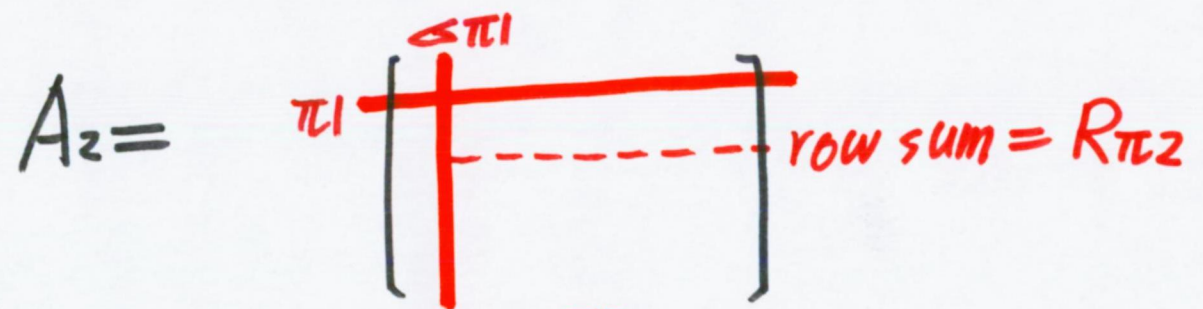
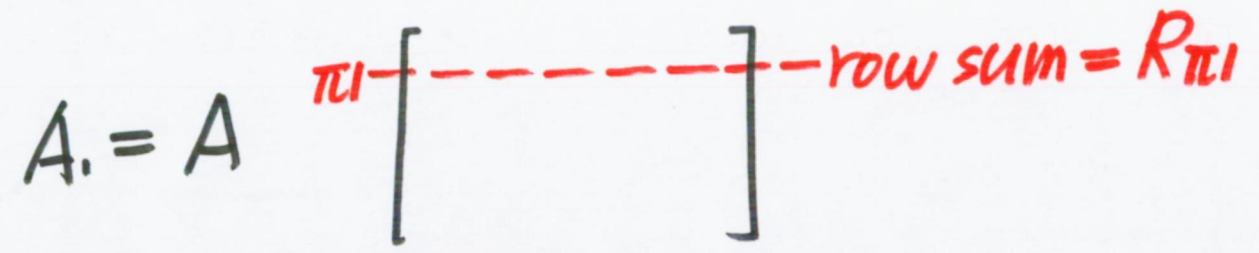
$$r_i \stackrel{\text{def}}{=} \sum_{j=1}^n a_{ij} \quad \text{row sum of } i\text{th row of } A$$

$$S \stackrel{\text{def}}{=} \{ \sigma \in S_n : a_{1\sigma_1} a_{2\sigma_2} \dots a_{n\sigma_n} = 1 \}$$

Sample space: Pick  $\sigma \in S$  and  $\pi \in S_n$  uniformly and independently.

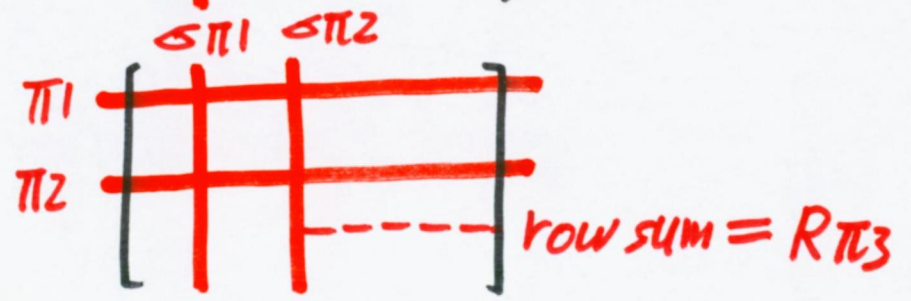
# Lazyman's permanent

$R_{\pi_1}, R_{\pi_2}, \dots, R_{\pi_n}$ :



Lazyman's permanent

$A_3 =$



**L** def  $R_{\pi_1} R_{\pi_2} \dots R_{\pi_n}$

Note: L tend to overestimate per(A).

3.

$$\text{per}(A) \leq e^{E[\ln L]}$$

**pf:**  $E(\ln L | \pi_i = i) = E\{E(\ln L | \pi_i = i, \sigma_i)\}$

$$= \sum_{j=1}^{r_i} E(\ln L | \pi_i = i, \sigma_i = j) P_r(\sigma_i = j)$$

w.l.o.g here we assume that

$a_{i1} = a_{i2} = \dots = a_{ir_i} = 1$  and

$a_{i r_i + 1} = \dots = a_{in} = 0$   $i \rightarrow [1 \dots r_i]$

$$\geq \sum_{j=1}^{r_i} \ln(r_i t_j) \frac{t_j}{|S|}, \text{ where } t_j = |\{\sigma \in S : \sigma_i = j\}|.$$

subpermanent

$$= (\ln r_i) + \sum_{j=1}^{r_i} \frac{r_i}{|S|} \frac{1}{r_i} (t_j \ln t_j), \quad \because |S| = t_1 + t_2 + \dots + t_{r_i}$$

$$\geq (\ln r_i) + \frac{r_i}{|S|} \left(\frac{t_1 + \dots + t_{r_i}}{r_i}\right) \ln\left(\frac{t_1 + \dots + t_{r_i}}{r_i}\right) \quad \because f(x) = x \ln x \text{ is convex.}$$

$$= (\ln r_i) + \ln\left(\frac{t_1 + \dots + t_{r_i}}{r_i}\right) = \ln \text{per}(A). \quad \because \text{per}(A) = |S|.$$

Note that  $E(\ln L) = E(E(\ln L | \pi_i)) \geq \ln \text{per}(A).$

Therefore  $e^{E(\ln L)} \geq \text{per}(A).$

**QED**

# Bregman's Thm (II)

Thm (1973)

$$\text{per}(A) \leq \prod_{1 \leq i \leq n} (r_i!)^{\frac{1}{r_i}}$$

pf:

$$\begin{aligned} e^{\mathbb{E} \ln R_{\pi i}} &= e^{\sum_{k=1}^{r_{\pi i}} \ln k \Pr(R_{\pi i} = k)} \\ &= e^{\sum_{k=1}^{r_{\pi i}} \ln k \frac{1}{r_{\pi i}}} \\ &= \left( \prod_{k=1}^{r_{\pi i}} k \right)^{\frac{1}{r_{\pi i}}} = (r_{\pi i}!)^{\frac{1}{r_{\pi i}}} \end{aligned}$$

$$\begin{aligned} e^{\mathbb{E}(\ln L)} &= e^{\mathbb{E}(\ln R_{\pi 1} R_{\pi 2} \cdots R_{\pi n})} \\ &= e^{\mathbb{E} \sum_{i=1}^n \ln R_{\pi i}} = \prod_{1 \leq i \leq n} e^{\mathbb{E} \ln R_{\pi i}} = \prod_{1 \leq i \leq n} (r_i!)^{\frac{1}{r_i}} \end{aligned}$$

QED