Brégman's Thm (I)

$A = [a_{ij}] \in M_{n\times n}[0,1]$

$\mathrm{per}(A) \overset{\text{def}}{=} \sum_{\sigma \in S_n} a_{1\sigma 1} a_{2\sigma 2} \cdots a_{n\sigma n}$

$r_i \overset{\text{def}}{=} \sum_{j=1}^{n} a_{ij}$ row sum of ith row of A

$S \overset{\text{def}}{=} \{ \sigma \in S_n : a_{1\sigma 1} a_{2\sigma 2} \cdots a_{n\sigma n} = 1 \}$

Sample space: Pick $\sigma \in S$ and $\pi \in S_n$ uniformly and independently.

Lazyman's permanent

$R_{\pi 1}, R_{\pi 2}, \cdots, R_{\pi n}$:

$$A_1 = A \quad \begin{array}{c|c|} \pi 1 & \text{- - - - - - -} \\ & \\ & \end{array} \text{ row sum} = R_{\pi 1}$$

$$A_2 = \quad \begin{array}{cc} & \sigma\pi 1 \\ \pi 1 & \left[\begin{array}{c} \;\; \\ \text{- - - - - -} \\ \;\; \end{array}\right] \end{array} \text{ row sum} = R_{\pi 2}$$

$$A_3 = \quad \begin{array}{cc} & \sigma\pi 1 \quad \sigma\pi 2 \\ \pi 1 & \left[\begin{array}{c} \;\; \\ \;\; \\ \text{- - - - -} \end{array}\right] \\ \pi 2 & \end{array} \text{ row sum} = R_{\pi 3}$$

Lazeman's permanent

$$L \stackrel{\text{def}}{=\!=} R_{\pi 1} R_{\pi 2} \cdots R_{\pi n}$$

Note: L tend to overestimate $per(A)$.

$$\mathrm{per}(A) \le e^{E[\ln L]}$$

pf: $\mathcal{E}(\ln L \mid \pi 1 = i) = \mathcal{E}\{\mathcal{E}(\ln L \mid \pi 1 = i, \sigma i)\}$

$= \sum_{j=1}^{r_i} \mathcal{E}(\ln L \mid \pi 1 = i, \sigma i = j)\, P_r(\sigma i = j)$

w.l.o.g here we assume that $a_{i1} = a_{i2} = \cdots = a_{ir_i} = 1$ and $a_{i\,r_i+1} = \cdots = a_{in} = 0$

$i \to \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\ge \sum_{j=1}^{r_i} \ln(r_i \overset{\text{subpermanent}}{t_j}) \frac{t_j}{|S|}$, where $t_j = |\{\sigma \in S : \sigma i = j\}|$.

$= (\ln r_i) + \sum_{j=1}^{r_i} \frac{r_i}{|S|} \frac{1}{r_i} (t_j \ln t_j)$, $\because |S| = t_1 + t_2 + \cdots + t_{r_i}$

$\ge (\ln r_i) + \frac{r_i}{|S|} \left(\frac{t_1 + \cdots + t_{r_i}}{r_i}\right) \ln \left(\frac{t_1 + \cdots + t_{r_i}}{r_i}\right)$ $\because f(x) = x \ln x$ is convex.

$= (\ln r_i) + \ln \left(\frac{t_1 + \cdots + t_{r_i}}{r_i}\right) = \ln \mathrm{per}(A)$. $\because \mathrm{per}(A) = |S|$.

Note that $\mathcal{E}(\ln L) = \mathcal{E}(\mathcal{E}(\ln L \mid \pi 1)) \ge \ln \mathrm{per}(A)$.

Therefore $e^{\mathcal{E}(\ln L)} \ge \mathrm{per}(A)$.

QED

Brégman's Thm (II)

Thm (1973)

$$\mathrm{per}(A) \leq \prod_{1\leq i\leq n} (r_i!)^{\frac{1}{r_i}}$$

pf:

$$e^{\mathcal{E}\ln R_{\pi i}} = e^{\sum_{k=1}^{r_{\pi i}} \ln k \, \Pr(R_{\pi i}=k)}$$

$$= e^{\sum_{k=1}^{r_{\pi i}} \ln k \frac{1}{r_{\pi i}}}$$

$$= \left(\prod_{i=1}^{r_{\pi i}} k\right)^{\frac{1}{r_{\pi i}}} = (r_{\pi i}!)^{\frac{1}{r_{\pi i}}}.$$

$$e^{\mathcal{E}(\ln L)} = e^{\mathcal{E}(\ln R_{\pi 1} R_{\pi 2} \cdots R_{\pi n})}$$

$$= e^{\mathcal{E}\sum_{i=1}^{n} \ln R_{\pi i}} = \prod_{1\leq i\leq n} e^{\mathcal{E}\ln R_{\pi i}} = \prod_{1\leq i\leq n} (r_i!)^{\frac{1}{r_i}}.$$

QED