The Sieve Formula

Thm let $A_1, \cdots, A_n$ be events of $(\Omega, \mathcal{F}, P)$.

Then $P(A_1 + A_2 + \cdots + A_n) = \sum_{j=1}^{n} (-1)^{j-1} \sum_{I \in [{n \atop j}]} P(\prod_{i \in I} A_i)$

pf: Let $\mathcal{F} = \{ B_1 \cdots B_k \overline{B_{k+1}}\, \overline{B_{k+2}} \cdots \overline{B_n} : 0 \le k \le n, \{B_1, \dots, B_n\} = \{A_1, \cdots, A_n\} \}$.

One can express each $\prod_{i \in I} A_i$ and $A_1 + A_2 + \cdots + A_n$ as the disjoint union of the corresponding atom in $\mathcal{F}$.

Let $B = A_1 A_2 \cdots A_k \overline{A}_{k+1} \overline{A}_{k+2} \cdots \overline{A}_n$.

- The coefficient of $P(B)$ on the LHS $= \begin{cases} 1 & \text{if } k \neq 0, \\ 0 & \text{if } k = 0. \end{cases}$
- The coefficient of $P(B)$ on the RHS is 0 if $k = 0$, and

$$\sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} = 1 - \sum_{j=0}^{k} (-1)^j \binom{k}{j} = 1 - (1-1)^k = 1 \quad \text{if } k \neq 0.$$

QED

Rényi's Thm

Thm Let $A_1, \cdots, A_n$ be any events, $B_i = f_i(A_1, \cdots, A_n)$ $i=1,\cdots,k$ polynomials in $A_1, \cdots, A_n$ and $c_1, \cdots, c_n$ reals. Then

$$\sum_{i=1}^{k} c_i P(B_i) \geq 0$$ holds for every $A_1, \cdots, A_n$,

provided it holds in those cases when

$P(A_j) = 1$ or 0 for $j = 1, 2, \cdots, n$.

pf: • B_i is the sum of atoms $A_1 \cdots A_\ell \bar{A}_{\ell+1} \cdots \bar{A}_n$.

• For an atom $\hat{\omega}$ define a new probability measure $\mathbb{P}$ s.t.

$\mathbb{P}(\hat{\omega}) = 1$ and $\mathbb{P}(\omega) = 0$ for any other atom ω.

let $\lambda_\omega \stackrel{\text{def}}{=}$ the coefficient of $\mathbb{P}(\omega)$ in $\sum_{i=1}^{k} c_i \mathbb{P}(B_i)$ for any atom ω.

Thus $\lambda_{\hat{\omega}} = \sum_{\omega \text{ is an atom}} \lambda_\omega \mathbb{P}(\omega) = \sum_{i=1}^{k} c_i \mathbb{P}(B_i) \geq 0$, by hypothesis.

Therefore $\lambda_\omega \geq 0$ for $\forall$ atom ω, and hence $\sum_{i=1}^{k} c_i P(B_i) \geq 0$. **QED**

Corollary Suppose we have "$\sum_{i=1}^{k} c_i P(B_i) = 0$" in Rényi's Thm. Then it holds for every choice of events $A_1, \dots, A_n$.

Pf: Apply Rényi's Thm in turn to the inequalities

$$\sum_{i=1}^{k} c_i P(B_i) \geq 0 \quad \text{and} \quad \sum_{i=1}^{k} (-c_i) P(B_i) \geq 0.$$

This thm now follows immediately.

QED

Corollary (inclusion-Exclusion formula)

$$P\left(\bigcup_{i=1}^{n} A_i\right) + \sum_{j=1}^{n} (-1)^j \sum_{|J|=j} P\left(\bigcap_{i\in J} A_i\right) = 0,$$ equivalently

$$P\left(\bigcup_{i=1}^{n} A_i\right) + \sum_{|J|\geq 1} (-1)^{|J|} P\left(\bigcap_{i\in J} A_i\right) = 0.$$

pf: Suppose $P(A_1) = P(A_2) = \cdots = P(A_l) = 1$ &

$P(A_{l+1}) = P(A_{l+2}) = \cdots = P(A_n) = 0$, where $l \geq 1$.

$$\text{LHS} = 1 + \sum_{1\leq j\leq l} (-1)^j \binom{l}{j}$$

$$= \sum_{j=0}^{l} (-1)^j \binom{l}{j} = (1-1)^l = 0 = \text{RHS}.$$

QED

Alternating Inequalities

Def: A sum $s=\sum_{k=1}^{n}(-1)^{k+1}x_k$ satisfies the alternating inequalities if

$$(-1)^{l}\left\{s-\sum_{k=1}^{l}(-1)^{k+1}x_k\right\}\geq 0,\text{ for every } l,\ l=0,1,2,\cdots,n.$$

Note: (1) $x_k\geq 0,\ \forall k$ (2) $\sum_{k=1}^{2m}(-1)^{k+1}x_k\leq s\leq\sum_{k=1}^{2m+1}(-1)^{k+1}x_k$.

Fact: $\sum_{k=0}^{m}(-1)^k\binom{n}{k}=(-1)^m\binom{n-1}{m}$.

pf: By induction on m. $\sum_{k=0}^{m}(-1)^k\binom{n}{k}=(-1)^{m-1}\binom{n-1}{m-1}+(-1)^m\binom{n}{m}=(-1)^m\binom{n-1}{m}$.

Note: $\sum_{r=k}^{n}(-1)^{r-k}\binom{r}{k}\sum_{J\in\left[{n \atop r}\right]}P_r\left(\bigcap_{j\in J}A_j\right)=\sum_{i=0}^{n-k}(-1)^i\binom{k+i}{i}\sum_{J\in\left[{n \atop k+i}\right]}P_r\left(\bigcap_{j\in J}A_j\right)$

Jordan's Formula & a strengthening of Bonferroni's Ineq.

Thm Let $A_1, \dots, A_n$ be events in a pro. space. Let $P_k = P_r\{$exactly k of the events A_i's occur$\}$. Then $P_k = \sum_{r=k}^{n} (-1)^{r-k} \binom{r}{k} \sum_{J \in [n \atop r]} P_r\left(\bigcap_{j \in J} A_j\right)$, and the sum satisfies the alternating inequalities.

$\bigcap_{j \in \phi} A_j = \Omega$

Pf: Suppose $P(A_1) = \cdots = P(A_\ell) = 1$ & $P(A_{\ell+1}) = \cdots = P(A_n) = 0$, where $\ell \geq 1$.

Then LHS $= 0$ if $\ell \neq k$, and 1 if $\ell = k$.

RHS $= 0$ if $\ell < k$,

RHS $= (-1)^{k-k}\binom{k}{k} P(A_1 \cap \cdots \cap A_k) = 1$ if $\ell = k$.

RHS $= \sum_{r=k}^{\ell} (-1)^{r-k} \binom{r}{k}\binom{\ell}{r} = \binom{\ell}{k} \sum_{r=k}^{\ell} (-1)^{r-k} \binom{\ell-k}{r-k} = \binom{\ell}{k}(1-1)^{\ell-k} = 0$,

if $\ell > k$.

Therefore LHS = RHS by Rényi's Thm.

Pf (continued)

Let $$\bigstar_m = (-1)^{m+1}\left\{ P_k - \sum_{r=k}^{k+m} (-1)^{r-k}\binom{r}{k} \sum_{J\in\left[{n \atop r}\right]} P\Big(\bigcap_{j\in J} A_j\Big) \right\}.$$

Then $$l<k \Rightarrow \bigstar_m = (-1)^{m+1} P_k = 0$$

$$l=k \Rightarrow \bigstar_m = (-1)^{m+1}\left\{ 1 - (-1)^{k-k}\binom{k}{k} 1 \right\} = 0$$

$$k<l\le k+m \Rightarrow \bigstar_m = (-1)^{m+1}\left\{ 0 - \sum_{r=k}^{l} (-1)^{r-k}\binom{r}{k}\binom{l}{r} \right\}$$

$$= (-1)^{m+1}\left\{ 0 - \binom{l}{k}\sum_{r=k}^{l} (-1)^{r-k}\binom{l-k}{r-k} \right\}$$

$$= (-1)^{m+1}\left\{ 0 - \binom{l}{k}(1-1)^{l-k} \right\} = 0.$$

$$k+m<l \Rightarrow \bigstar_m = (-1)^{m+1}\left\{ 0 - \sum_{r=k}^{k+m} (-1)^{r-k}\binom{r}{k}\binom{l}{r} \right\}$$

$$= (-1)^{m}\binom{l}{k}\sum_{i=0}^{m} (-1)^{i}\binom{l-k}{i}$$

$$= (-1)^{m}\binom{l}{k}(-1)^{m}\binom{l-k-1}{m} = \binom{l}{k}\binom{l-k-1}{m} \ge 0$$

Therefore $\bigstar_m \ge 0$ for every m.

QED

A direct proof of Jordan's formula

Thm If $A_1, \cdots, A_n \subseteq S$, S is a finite set. Then

$$\#\{s \in S : s \text{ lies in exactly } k \text{ of the sets } A_i\} = \sum_{r=k}^{n} (-1)^{r-k} \binom{r}{k} \sum_{J \in \binom{[n]}{r}} \left| \bigcap_{j \in J} A_j \right|.$$

pf: Fix $J \in \binom{[n]}{k}$. let $B = \bigcap_{j \in J} A_j$ and $\bar{J} = [n] \setminus J$.

$$\#\{s \in S : s \in B \text{ but } s \notin \bigcup_{i \in \bar{J}} A_i\} = |B| - \left| \bigcup_{i \in \bar{J}} (A_i B) \right|$$

$$= |B| - \sum_{i \in \bar{J}} |A_i B| + \sum_{i, j \in \bar{J},\ i<j} |A_i A_j B| - \sum_{i,j,k \in \bar{J},\ i<j<k} |A_i A_j A_k B| + \cdots = \sum_{J \subseteq I \subseteq [n]} (-1)^{|I|-k} \left| \bigcap_{i \in I} A_i \right|$$

Thus

$$\text{LHS} = \sum_{J \in \binom{[n]}{k}} \sum_{J \subseteq I \subseteq [n]} (-1)^{|I|-k} \left| \bigcap_{i \in I} A_i \right|$$

$$= \sum_{r=k}^{n} \sum_{I \in \binom{[n]}{r}} \binom{r}{k} (-1)^{r-k} \left| \bigcap_{i \in I} A_i \right| = \text{RHS}.$$

QED

Bonferroni's Inequalities

Thm let $A_1, \ldots, A_n$ be events in a pro. space. Then for any k and m

$$\sum_{i=0}^{2m+1} (-1)^i \binom{k+i}{i} \sigma_i \leq P_r(\text{exactly } k \text{ of the } A_i\text{'s occur}) \leq \sum_{i=0}^{2m} (-1)^i \binom{k+i}{i} \sigma_i,$$

where $\sigma_i = \sum_{J \in [{n \atop k+i}]} P_r\left(\bigcap_{j \in J} A_j\right)$ and $\sigma_0 = 1$.

Corollary

$$\bar{\sigma}_1 - \bar{\sigma}_2 + \bar{\sigma}_3 - \cdots - \bar{\sigma}_{2m} \leq P_r\left(\bigcup_{i=1}^{n} A_i\right) \leq \bar{\sigma}_1 - \bar{\sigma}_2 + \bar{\sigma}_3 - \cdots + \bar{\sigma}_{2m+1}$$

where $\bar{\sigma}_i = \sum_{J \in [{n \atop i}]} P_r\left(\bigcap_{j \in J} A_j\right)$.

Convergence of pdf's

Thm let $p = (\ln n + x)/n$ and $V(G_{n,p}) = [n]$.

Then $\lim_{n\to\infty} \Pr(G_{n,p}$ has exactly k isolated vertices$) = \dfrac{e^{-\lambda}\lambda^k}{k!}$, where $\lambda = e^{-x}$.

pf: let $A_i = \{ i$ is an isolated vertex in $G_{n,p}\}$.

$P($exactly k of the A_i's occur$) \leq \sum_{i=0}^{2m} (-1)^i \binom{k+i}{i} \sigma_i$ ★

Note that $\sigma_i = \sum_{J \in \binom{[n]}{r}} \Pr(J$ are isolated vertices in $G_{n,p})$, where $r = k+i$.

$$= \binom{n}{r}(1-p)^{r(n-r)+\binom{r}{2}} = \frac{n(n-1)\cdots(n-r+1)}{r!}(1-p)^{nr}(1-p)^{-r^2+\binom{r}{2}}$$

$$= \frac{n^r(1-p)^{nr}}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)(1-p)^{-r^2+\binom{r}{2}}$$

$$\underset{n\to\infty}{\sim} \frac{[n(1-p)^n]^r}{r!} = \frac{\left[n\left(1-\frac{\ln n + x}{n}\right)^n\right]^r}{r!} \underset{\substack{e^{-x} \doteq 1-x \\ \infty \quad x \doteq 0}}{\sim} \frac{[n e^{-\ln n - x}]^r}{r!} = \frac{\lambda^r}{r!}$$

Thus $★ \underset{n\to\infty}{=} \dfrac{\lambda^k}{k!} \sum_{i=0}^{2m} \dfrac{(-\lambda)^i}{i!}$ for any m. therefore LHS $\leq \dfrac{e^{-\lambda}\lambda^k}{k!}$.

And the reverse inequality follows in the similar way.

QED