Practice Problem

Example A random permutation has 1 fixed point on the average.

pf: Let $X_i(\sigma) = \begin{cases} 1 & \text{if } \sigma i = i \\ 0 & \text{o.w.} \end{cases}$, where σ is a random permutation on $\{1, 2, \cdots, n\}$.

Let X = the # of loops in σ.

$$\mathcal{E}X = \sum_{i=1}^{n} \mathcal{E}X_i = \sum_{i=1}^{n} \frac{(n-1)!}{n!} = 1$$

QED

Hamiltonian Paths

Thm the first use of the probabilistic method (Szele 1943) $\exists$ a tournament on n vertices that has at least $\frac{n!}{2^{n-1}}$ Hamiltonian dipaths.

Pf: $X \overset{\text{def}}{=\!=}$ the # of Hamiltonian dipaths.

$$X_\sigma \overset{\text{def}}{=\!=} \begin{cases} 1 & \text{if } \sigma \text{ is a Hamiltonian dipath.} \\ 0 & \text{o.w.} \end{cases}$$

$$\mathcal{E}X = \mathcal{E} \sum_{\sigma \in S_n} X_\sigma$$

$$= \sum_{\sigma \in S_n} P_r\{\sigma_1 \sigma_2 \ldots \sigma_n \text{ is a Ham. dipath})$$

$$= n! \, \frac{2^{\binom{n}{2}-(n-1)}}{2^{\binom{n}{2}}} = \frac{n!}{2^{n-1}}.$$

QED

Two Quickies

Thm $\exists$ a $\overset{\text{edge}}{2}$-coloring of K_n with at most $\binom{n}{a} 2^{1-\binom{a}{2}}$ monochromatic K_a.

Pf: $X_S \overset{\text{def}}{=} \begin{cases} 1 & \text{if } S \text{ is monochromatic} \\ 0 & \text{o.w.} \end{cases}$

$$\mathcal{E} \sum_{S \in \left[{n \atop a}\right]} X_S = \sum_{S \in \left[{n \atop a}\right]} P(X_S = 1) = \binom{n}{a} 2 \cdot 2^{-\binom{a}{2}}.$$

QED

Thm $\exists$ a $\overset{\text{edge}}{2}$-coloring of $K_{m,n}$ with at most $\binom{m}{a}\binom{n}{b} 2^{1-ab}$ monochromatic $K_{a,b}$.

Pf: $X_{S,T} \overset{\text{def}}{=} \begin{cases} 1 & \text{if } [S,T] \text{ is monochromatic} \\ 0 & \text{o.w.} \end{cases}$

$$\mathcal{E} \sum_{S \in \left[{m \atop a}\right], T \in \left[{n \atop b}\right]} X_{S,T} = \binom{m}{a}\binom{n}{b} 2 \cdot 2^{-ab}.$$

QED

Splitting Graphs I.

Thm Any graph with m edges contains a bipartite subgraph with at least $\frac{m}{2}$ edges.

pf: Let S be a random vertex subset.

Let $X_{uv}=\begin{cases}1 & \text{if } |\{u,v\}\cap S|=1\\ 0 & \text{o.w.}\end{cases}$

$$\mathcal{E}\sum_{uv\in E(G)}X_{uv}=\sum_{uv\in E(G)}\mathcal{E}X_{uv}=\sum_{uv\in E(G)}1-P_r\{\{u,v\}\subseteq S \text{ or } \{u,v\}\cap S=\emptyset\}$$

$$=\frac{m}{2}$$

QED

Splitting Graphs II

Thm If G has $\begin{matrix}2n\\2n+1\end{matrix}$ vertices and e edges then it contains a bipartite subgraph with at least

$\begin{matrix}en/(2n-1)\\e(n+1)/(2n+1)\end{matrix}$ edges.

pf: Let $\Omega = \genfrac{[}{]}{0pt}{}{2n}{n}$ with $P(\omega) = 1/|\Omega|$.

Let $X_{xy}^{(\omega)} = \begin{cases} 1 & \text{if } |\{x,y\}\cap\omega| = 1 \\ 0 & \text{o.w.} \end{cases}$, where $xy \in E(G)$.

$$\mathcal{E}\sum_{x\sim y} X_{xy} = e\,P(\{\omega : |\overset{\text{where } xy\in E(G)}{\{x,y\}}\cap\omega| = 1\}) = e\,\frac{\binom{2n}{n} - \binom{2n-2}{n} - \binom{2n-2}{n-2}}{\binom{2n}{n}} = \frac{en}{2n-1}.$$

note: If $\Omega = \genfrac{[}{]}{0pt}{}{2n+1}{n}$ with $P(\omega) = 1/|\Omega|$ then

$$P(\{\omega : |\{x.y\}\cap\omega| = 1) = \frac{n+1}{2n+1}.$$

QED

Splitting Graphs III

Thm let $V = V_1 + \cdots + V_k$, $|V_i| = n$, $\forall i$. Then $\exists\ S \subseteq V$ for which

$$\left| \# \left\{ E \in \left[{S \atop k} \right] : |E \cap V_i| = 1,\ \forall i \right\} - \# \left\{ E \in \left[{S \atop k} \right] : |E \cap V_i| \neq 1 \text{ for some } i \right\} \right|$$

$\geq c_k n^k$ for some constant $c_k > 0$ independent of n.

pf: Let S be a random vertex subset s.t. $P(v \in S) = p_i$, $\forall v \in V_i$ (mutually independent)

For $E \in \left[{V \atop k} \right]$, $h(E) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } E \text{ is crossing w.r.t. } V_1, \cdots, V_k \\ -1 & \text{o.w.} \end{cases}$, $X_E \overset{\text{def}}{=} \begin{cases} h(E) & \text{if } E \subseteq S \\ 0 & \text{o.w.} \end{cases}$

$$\mathcal{E} \left| \sum_{E \in \left[{V \atop k} \right]} X_E \right| \geq \left| \sum_E \mathcal{E} X_E \right| = \left| \sum_{E \in \left[{V \atop k} \right]} h(E) P(E \subseteq S) \right|$$

$$= \left| \sum_{a_1 + \cdots + a_k = k} \underbrace{\left(\sum_{\substack{|E \cap V_i| = a_i \\ i = 1, \cdots, k;\ E \in \left[{V \atop k} \right]}} h(E) \right)} p_1^{a_1} \cdots p_k^{a_k} \right| = n^k \left| f(p_1, p_2, \cdots, p_k) \right| \underset{\uparrow}{\geq} n^k c_k \quad (\exists \text{ such } p_1, \cdots, p_k \in [0,1]$$

P15 Lemma 2.2.4 P15

where $f(x_1, \cdots, x_k) = {}^{\star}\sum_{a_1 + \cdots + a_k = k} \frac{\star}{n^k} x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$ is a homogeneous polynomials of degree k with coefficient $(x_1 x_2 \cdots x_k) = 1$ and

$$|\text{coefficient}(x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k})| \leq \frac{\binom{n}{a_1} \binom{n}{a_2} \cdots \binom{n}{a_k}}{n^k} \leq 1, \quad x_1, \cdots, x_k \in [0,1].$$

QED

Balancing Vectors I

Thm Suppose $v_1, \cdots, v_n \in \mathbb{R}^n$ with $|v_i| = 1$. Then $\exists\ \varepsilon_1, \cdots \varepsilon_n \in \{\pm 1\}$ s.t.

$$|\varepsilon_1 v_1 + \varepsilon_2 v_2 + \cdots + \varepsilon_n v_n| \underset{(\geq)}{\leq} \sqrt{n}$$

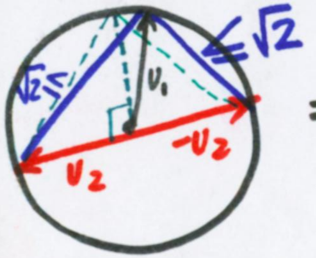
column vector

pf: $X_1, \cdots, X_n \overset{\text{iid}}{\longrightarrow} P(X_1 = \pm 1) = \frac{1}{2}$. $\quad Y \overset{\text{def}}{=} |\sum_{i=1}^{n} X_i v_i|$

$$\mathcal{E} Y^2 = \mathcal{E}\left([\sum_{i=1}^{n} X_i v_i]^t [\sum_{i=1}^{n} X_i v_i] \right) = \sum_{1 \leq i, j \leq n} \mathcal{E}(X_i X_j)\, v_i^t v_j = \sum_{i=1}^{n} \mathcal{E}(X_i^2)\, |v_i|^2 = n.$$

QED

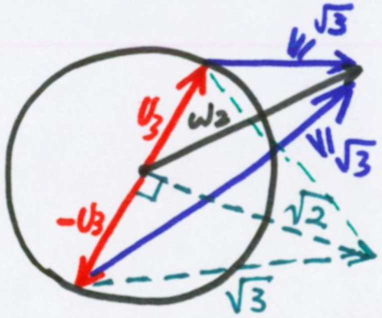
<u>nonprobabilistic proof</u>: $\boxed{\text{Idea}}$ let $\omega_i = \varepsilon_1 v_1 + \cdots + \varepsilon_i v_i$. $\quad i = 1, 2, \cdots, n$.



$\Rightarrow |v_1 + v_2| \leq \sqrt{2}$

let $\omega_2 = v_1 + v_2$

i.e. $\varepsilon_1 = 1$, $\varepsilon_2 = 1$.



$\Rightarrow |\omega_2 - v_3| \leq \sqrt{3}$

$\Rightarrow |v_1 + v_2 - v_3| \leq \sqrt{3}$

There we conclude that $\exists\ \varepsilon_1, \cdots, \varepsilon_n \in \{\pm 1\}$ s.t.

$$|\varepsilon_1 v_1 + \cdots + \varepsilon_i v_i| \leq \sqrt{i} \qquad i = 1, 2, \cdots, n.$$

QED

Balancing Vectors II

Thm Suppose $v_1, \dots, v_n \in \mathbb{R}^n$ with $|v_i| \le 1$. $p_1, \dots, p_n$ are fixed values in $[0,1]$.
Then $\exists\ \varepsilon_1, \dots, \varepsilon_n \in \{0,1\}$ s.t. $\left| \sum_{i=1}^n (\varepsilon_i - p_i) v_i \right| \le \sqrt{n}/2$

pf: let $Y = \left| \underbrace{\sum_{i=1}^n (X_i - p_i) v_i}_{*} \right|$, where $X_1, X_2, \dots, X_n$ are independent rvs $X_i \sim B(1, p_i)$.

$$\mathcal{E} Y^2 = \mathcal{E}(*^t *) = \sum_{i=1}^n |v_i|^2 \mathcal{E}(X_i - p_i)^2 = \sum_{i=1}^n p_i (1-p_i) |v_i|^2 \le n/4.$$

QED

non probabilistic proof: We claim that $\exists\ \varepsilon_1, \dots \varepsilon_n \in \{0,1\}$ s.t. $\left| \sum_{i=1}^s (\varepsilon_i - p_i) v_i \right| \le \frac{\sqrt{s}}{2}$ for $s = 1, 2, \dots, n$. Indeed, let $\omega_s = \sum_{i=1}^s (\varepsilon_i - p_i) v_i$. we have $|\omega_1|^2 \le |\varepsilon_1 - p_1|^2$. If $p_1 \le \frac{1}{2}$ then choose $\varepsilon_1 = 0$. If $p_1 \ge \frac{1}{2}$ then choose $\varepsilon_1 = 1$. Thus $\exists\ \varepsilon_1 \in \{0,1\}$ s.t. $|\omega_1|^2 \le 1/4$. By induction on s, let $X \sim B(1, p_i)$ and $f(X) = |\omega_{i-1} + (X - p_i) v_i|^2$.

$$\mathcal{E} f(X) = \mathcal{E} \left\{ |\omega_{i-1}|^2 + 2(X - p_i)\, \omega_{i-1}^t v_i + |v_i|^2 (X - p_i)^2 \right\}$$
$$= |\omega_{i-1}|^2 + |v_i|^2 \operatorname{Var} X = |\omega_{i-1}|^2 + |v_i|^2 p_i (1 - p_i) \le \frac{i-1}{4} + \frac{1}{4} = \frac{i}{4}.$$

QED

Unbalancing Lights

Thm Suppose $A \in \mathcal{M}_{n\times n}[\pm 1]$. Then $\exists\ x, y \in \mathcal{M}_{n\times 1}[\pm 1]$ s.t.

$$x^t A y \geq \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{\frac{3}{2}}.$$

pf: To show $\exists\ y_1, \dots, y_n \in \{\pm 1\}$ s.t. $\sum_{i=1}^n \left|\sum_{j=1}^n a_{ij} y_j\right| \geq \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{3/2}$.

Let $Y_1, \dots, Y_n \overset{iid}{\sim} P(Y_1 = 1) = P(Y_1 = -1) = 1/2$. Let $S_n^i = \sum_{j=1}^n a_{ij} Y_j$. (note: $a_{i1}Y_1, \dots, a_{in}Y_n \overset{iid}{\sim}$)

Then CLT $\Rightarrow$

$$\frac{S_n^i - \mathcal{E} S_n^i}{\sqrt{Var\, S_n^i}} \xrightarrow{\text{converges in dis.}} N(0,1) \Rightarrow \frac{S_n^i - 0}{\sqrt{n}} \to N(0,1) \quad \left(\because \mathcal{E}\, a_{ij} Y_j = 0,\ \mathcal{E}\, Y_j^2 = 1\right)$$

Therefore

$$\mathcal{E}|S_n^i| \sim \int_{\mathbb{R}} \sqrt{n}\, |t| \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \frac{2\sqrt{n}}{\sqrt{2\pi}} \int_0^\infty t e^{-\frac{t^2}{3}} dt = \sqrt{\frac{2n}{\pi}} \quad \text{as } n \to \infty$$

$$\mathcal{E} \sum_{i=1}^n \left|\sum_{j=1}^n a_{ij} Y_j\right| = \sum_{i=1}^n \mathcal{E}|S_n^i| = \sum_{i=1}^n \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{\frac{1}{2}} = \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{\frac{3}{2}}.$$

So we can choose x_i with the same sign as $\sum_{j=1}^n a_{ij} y_j$ to get

$$\sum_{i=1}^n \sum_{j=1}^n x_i a_{ij} y_j = \sum_{i=1}^n x_i \left(\sum_{j=1}^n a_{ij} y_j\right) = \sum_{i=1}^n \left|\sum_{j=1}^n a_{ij} y_j\right| \geq \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{\frac{3}{2}}.$$

QED

$\alpha(G)$

Thm $\alpha(G) \geq \sum_v \frac{1}{d_v+1}$

pf (Method 1) let π be a random permutation of $V=\{1,2,\cdots,n\}$

Let $I=\{v \in V : \pi(v) > \pi(x) \text{ for any } x \in N(v)\}$

$$I_v = \begin{cases} 1 & \text{if } v \in I \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} \mathcal{E}|I| &= \sum_{v\in V} \mathcal{E} I_v \\ &= \sum_{v\in V} P_r\{\pi(v) > \pi(x) \text{ for any } x \in N(v)\} \\ &= \sum_{v\in V} \frac{\binom{n}{d_v+1}(n-d_v-1)!\, d_v!}{n!} = \sum_v \frac{1}{d_v+1} \end{aligned}$$

QED (Method 1)

pf (Method 2) (Wei 1981, Griggs 1983)

Idea: 由 G 中不斷拿走現存 vertex 中 degree 最小的 vertex x 及其鄰居 i.e. delete $N[x]$.

$$G \longrightarrow \sum_{v\in G} \frac{1}{d_G(v)+1} \quad \star$$

$$G' = G - N[x] \longrightarrow \sum_{v\in G'} \frac{1}{d_{G'}(v)+1} \quad D$$

$$\vdots \qquad \vdots$$

Claim: If $\star > D$ then $\star - D \leq 1$

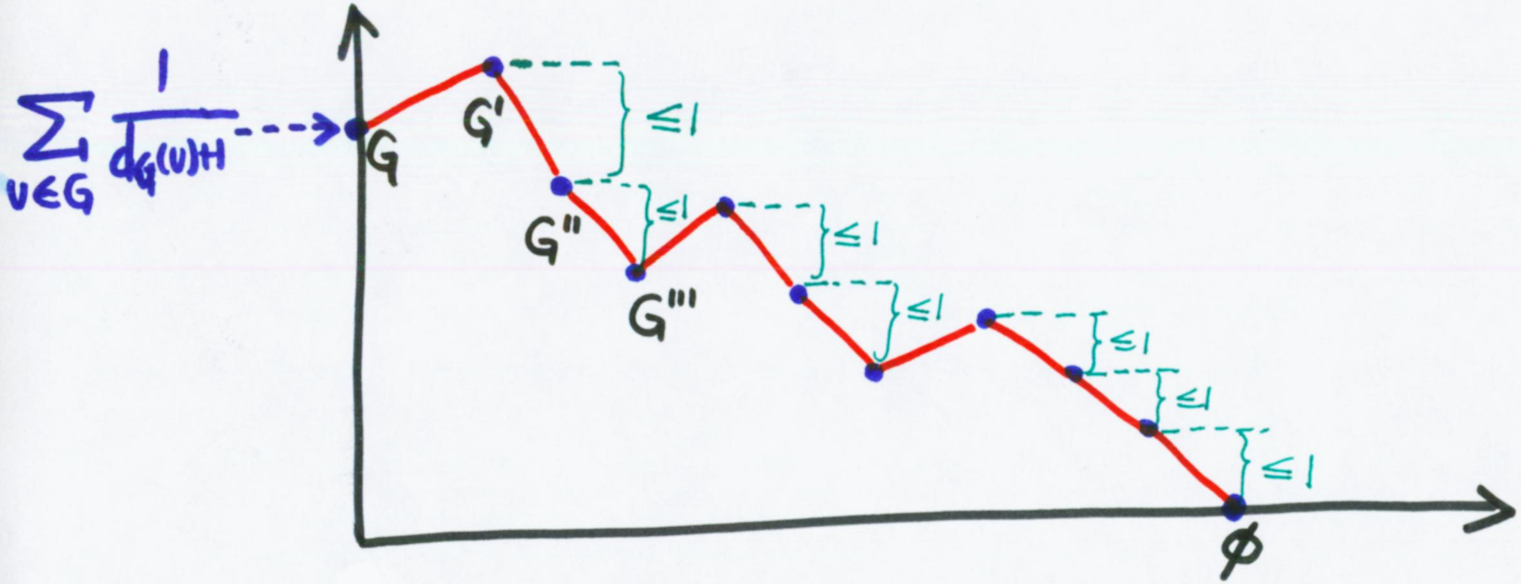
pf: If $d_G(x) = \delta(G)$, let $N_G(x) = \{v_1, v_2, \cdots, v_\ell\}$, then

$$\star - D \leq \frac{1}{d_G(v_1)+1} + \cdots + \frac{1}{d_G(v_\ell)+1} + \frac{1}{d_G(x)+1}$$

$$\leq \frac{d_G(x)+1}{d_G(x)+1} = 1$$

Pf (continued)

由下面圖示可知, 去掉的點數目 (i.e. 去掉步驟做的次數) 至少 $\geq \sum_{v\in G} \frac{1}{d_G(v)+1}$



Remark: see J. R. Griggs JCT Ser. B 34, 22-39 (1983)

$\alpha \ge \frac{\nu}{\bar{d}+1}$

Thm $\alpha(G) \ge \frac{\nu_G}{\bar{d}+1}$, where $\bar{d} = \frac{2e_G}{\nu_G}$ the average degree of G

pf: **Claim** If $x_1, \cdots, x_k > 0$ then $\frac{1}{x_1} + \cdots + \frac{1}{x_k} \ge \frac{k^2}{x_1+x_2+\cdots x_k}$

pf: $x, y > 0 \Rightarrow \frac{y}{x} + \frac{x}{y} \ge 2$

So $\sum_{1 \le i < j \le k} \left(\frac{x_i}{x_j} + \frac{x_j}{x_i} \right) \ge \binom{k}{2} 2 = k(k-1)$

$$\Rightarrow \frac{x_2+x_3+\cdots+x_k}{x_1} + \frac{x_1+x_3+\cdots+x_k}{x_2} + \cdots + \frac{x_1+x_2+\cdots+x_{k-1}}{x_k} \ge k(k-1)$$

$\Rightarrow$ Done! QED of Claim

$$\alpha(G) \ge \sum_v \frac{1}{d_v+1} \ge \frac{\nu_G^2}{\sum_v (d_v+1)} = \frac{\nu_G^2}{\nu_G + 2e_G}$$

QED

Turán's Lower Bound

Thm $\alpha(G) \geq \dfrac{|V(G)|}{1+\bar{d}}$ ← average degree of G.

pf: let $n = |V(G)|$ and $f(x) = \frac{1}{1+x}$.

degree of v

$$\alpha(G) \geq n \sum_v \frac{1}{n} f(d(v))$$

Jensen's Inequality →

$$\geq n \, f\Big(\sum_v \frac{1}{n} d(v)\Big)$$

$$= n \frac{1}{1+\bar{d}}$$

QED.

Convex Functions and Jensen's Inequality

A real-valued function f is *convex* on an interval I if and only if

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \tag{1}$$

for all $a, b \in I$ and $0 \leq t \leq 1$. This just says that a function is convex if the graph of the function lies below its secants. See pages 2 through 5 of Bjorn Poonen's paper, distributed at his talk on inequalities, for a discussion of convex functions and inequalities for convex functions. A number of common functions that are convex are also listed. Among those listed are $-\ln x$ on $(0,\infty)$, $-\sin x$ on $[0,\pi]$, $-\cos x$ on $[-\pi/2, \pi/2]$ and $\tan x$ on $[0, \pi/2]$. To avoid the negative signs a complementary concept is defined. A real-valued function f is *concave* on an interval I if and only if

$$f(ta + (1-t)b) \geq tf(a) + (1-t)f(b) \tag{2}$$

for all $a, b \in I$ and $0 \leq t \leq 1$. Therefore f is convex iff $-f$ is concave. If you are familiar with derivatives then the following theorem about twice differentiable functions provides a way of telling if such a function is convex.

Theorem 3 If $f''(x) \geq 0$ for all $x \in I$, then f is convex on I.

Inequality (1) can be generalized to a convex function f with three variables x_1, x_2, x_3 with weights t_1, t_2, t_3, respectively, such that $t_1 + t_2 + t_3 = 1$. Note that $t_2 + t_3 = 1 - t_1$. In this manner the three variable case can be transformed into the two variable case as follows.

$$\begin{aligned}
f(t_1x_1 + t_2x_2 + t_3x_3) &= f\left(t_1x_1 + (1-t_1)\frac{t_2x_2 + t_3x_3}{t_2 + t_3}\right) \\
&\leq t_1 f(x_1) + (1-t_1) f\left(\frac{t_2x_2 + t_3x_3}{t_2 + t_3}\right) \\
&= t_1 f(x_1) + (1-t_1) f\left(\frac{t_2}{t_2 + t_3}x_2 + \frac{t_3}{t_2 + t_3}x_3\right) \\
&\leq t_1 f(x_1) + (t_2 + t_3)\left(\frac{t_2}{t_2 + t_3}f(x_2) + \frac{t_3}{t_2 + t_3}f(x_3)\right) \\
&= t_1 f(x_1) + t_2 f(x_2) + t_3 f(x_3).
\end{aligned}$$

This process can be continued to produce an n variable version which is due to J.L.W.V. Jensen. It can be easily proved by mathematical induction using the above technique. Write your own proof and compare with the one given here. It will give you some good practice manipulating sigma notation.

Theorem 4 (Jensen's Inequality 1906) Let f be a convex function on the interval I. If $x_1, x_2, \ldots, a_n \in I$ and $t_1, t_2, \ldots, t_n$ are nonnegative real numbers such that $t_1 + t_2 + \ldots + t_n = 1$, then

$$f(\sum_{i=1}^{n} t_i x_i) \leq \sum_{i=1}^{n} t_i f(x_i).$$

Proof by induction: The case for $n = 2$ is true by the definition of convex. Assume the relation holds for n, then we have

$$\begin{aligned}
f\left(\sum_{i=1}^{n+1} t_i x_i\right) &= f\left(\sum_{i=1}^{n} t_i x_i + t_{n+1} x_{n+1}\right) = f\left(t_{n+1} x_{n+1} + (1 - t_{n+1}) \frac{1}{1 - t_{n+1}} \sum_{i=1}^{n} t_i x_i\right) \\
&\leq t_{n+1} f(x_{n+1}) + (1 - t_{n+1}) f\left(\frac{1}{1 - t_{n+1}} \sum_{i=1}^{n} t_i x_i\right) \\
&= t_{n+1} f(x_{n+1}) + (1 - t_{n+1}) f\left(\sum_{i=1}^{n} \frac{t_i}{1 - t_{n+1}} x_i\right) \\
&\leq t_{n+1} f(x_{n+1}) + (1 - t_{n+1}) \sum_{i=1}^{n} \frac{t_i}{1 - t_{n+1}} f(x_i) \\
&= \sum_{i=1}^{n} t_i f(x_i) + t_{n+1} f(x_{n+1}) \\
&= \sum_{i=1}^{n+1} t_i f(x_i).
\end{aligned}$$

Thus showing that the assumption implies that the relation holds for $n + 1$ and by the principle of Mathematical Induction holds for all natural numbers.

An easy consequence of Jensen's theorem is the following proof of the arithmetic mean-geometric mean inequality. (Problem 13 from Bjorn's paper)

Theorem 5 (AM-GM Inequality) If $x_1, x_2, \ldots, x_n \geq 0$ then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}.$$

Proof. Since $-\ln x$ is convex then $\ln x$ is concave. By Jensen's theorem we have

$$\begin{aligned}\ln\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) &\geq \frac{\ln x_1 + \ln x_2 + \cdots + \ln x_n}{n} \\ &= \frac{1}{n}\ln(x_1 x_2 \cdots x_n) \\ &= \ln[(x_1 x_2 \cdots x_n)^{\frac{1}{n}}]\end{aligned}$$

Since $\ln x$ is monotonic increasing ($f'(x) = \frac{1}{x} > 0$) for $x > 0$ we have

$\frac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}.$

The proof of Jensen's Inequality does not address the specification of the cases of equality. It can be shown that strict inequality exists unless all of the x_i are equal or f is linear on an interval containing all of the x_i .
