Ramsey Numbers

$R(G,H) \overset{\text{def}}{=} \min\{ n :$ every red-blue edge coloring of K_n contains a red G or a blue $H \}$

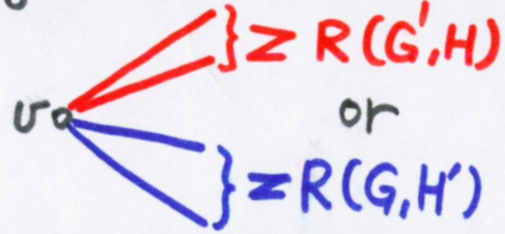
$R(k,l) \overset{\text{def}}{=} R(K_k, K_l)$ Facts 1. $r(K_1,G)=1$ 2. $r(K_2,G)=|V(G)|$.

obtained from G by deleting one vertex

Thm $\nu_G, \nu_H \geq 2$. Then $R(G,H) \leq R(G',H)+R(G,H')$. Moreover if both $R(G',H)$ & $R(G,H')$ are even then $R(G,H) < R(G',H)+R(G,H')$.

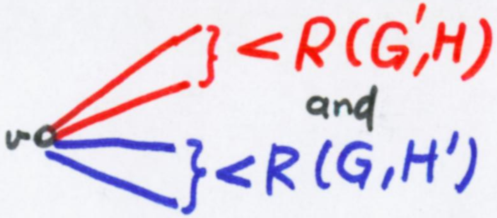
Pf: Let $n = R(G',H)+R(G,H')$.

(1) red-blue edge color K_n to get



Thus $R(G,H) \leq n$.

(2) Assume $R(G,H)=n$. So ∃ a red-blue edge colo of K_{n-1} s.t. no red G & no blue H. For every vertex v in K_{n-1}



Thus $= R(G',H)-1$ $= R(G,H')-1$.

And $|\text{red edges}| = \frac{(n-1)[R(G',H)-1]}{2}$ a contradiction since n and $R(G',H)$ are even.

QED

Bounding $R(K,\ell)$ without alternation method

Thm Given k and ℓ. Suppose we may choose $n \in \mathbb{Z}^+$ and $p \in [0,1]$ s.t. $\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} < 1$. Then $R(k,\ell) > n$.

Pf: To show ∃ an red-blue edge coloring of K_n with no red K_k and blue K_ℓ. Color edges of K_n red with probability p and blue with probability $(1-p)$.

$A \overset{\text{def}}{=} \{G_{n,p}$ contains a $K_k\}$. $B \overset{\text{def}}{=} \{G_{n,p}$ contains a $\overline{K_\ell}\}$

$$P(\bar{A} \cap \bar{B}) \geq 1 - \big(P(A) + P(B)\big) = 1 - \Big(P\Big(\bigcup_{S \in \left[{n \atop k}\right]} \{S \text{ induces a } K_k \text{ in } G_{n,p}\}\Big) + P(B)\Big)$$

$$\geq 1 - \sum_{S \in \left[{n \atop k}\right]} p^{\binom{k}{2}} - P(B) \geq 1 - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{\ell}(1-p)^{\binom{\ell}{2}}.$$

QED

Corollary: $R(k,k) \geq \frac{k}{e\sqrt{2}} 2^{k/2}$.

Pf: To find n s.t. $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$. Note that Stirling approximation says $\sqrt{2\pi K}\left(\frac{K}{e}\right)^K \leq K! \leq \sqrt{2\pi K}\left(\frac{K}{e}\right)^K e^{\frac{1}{12K}}$. Thus $\binom{n}{K} \leq \frac{n^K}{K!} \leq \frac{1}{\sqrt{2\pi K}}\left(\frac{en}{K}\right)^K$ and hence

$$\binom{n}{K} 2^{1-\binom{K}{2}} \leq \frac{2}{\sqrt{2\pi K}} \underbrace{\left(\frac{ne}{K}\right)^K 2^{-\binom{K}{2}}}_{*}.$$

Note that $* \leq 1 \Leftrightarrow \frac{ne}{K} 2^{-\frac{K-1}{2}} \leq 1 \Leftrightarrow n \leq \frac{K}{e\sqrt{2}} 2^{k/2}$ choose $n = \left\lfloor \frac{K}{e\sqrt{2}} 2^{K/2} \right\rfloor$, giving $R(K,K) > n$.

QED

Bounding $R(k,l)$ with alternation method

Thm For $n\in\mathbb{Z}^+$ and $p\in[0,1]$, $R(k,l) > n-\binom{n}{k}p^{\binom{k}{2}}-\binom{n}{l}(1-p)^{\binom{l}{2}}$

Pf: To show $\exists$ a graph of order n with at most $\binom{n}{k}p^{\binom{k}{2}}$ induced K_k and $\binom{n}{l}(1-p)^{\binom{l}{2}}$ induced $\overline{K_l}$. Deleting a single vertex in K_k (resp. $\overline{K_l}$) will destroy K_k (resp. $\overline{K_l}$). Consider $\mathcal{G}_{n,p}$. $X_R(\omega)\overset{def}{=}\begin{cases}1 & \text{if } \omega[R]\cong K_k\\ 0 & \text{o.w.}\end{cases}$; $Y_B(\omega)\overset{def}{=}\begin{cases}1 & \text{if } \omega[B]\cong \overline{K_l}\\ 0 & \text{o.w.}\end{cases}$

$Z\overset{def}{=} n-\sum_{R\in\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]}X_R-\sum_{B\in\left[\begin{smallmatrix}n\\l\end{smallmatrix}\right]}X_B$. $\mathcal{E}Z= n-\binom{n}{k}p^{\binom{k}{2}}-\binom{n}{l}(1-p)^{\binom{l}{2}}$.

QED

Corollary $R(k,k)\ge\frac{k}{e}(1+o(1))\,2^{k/2}$.

Pf:

$$R(k,k) > n-2\binom{n}{k}2^{-\binom{k}{2}} > n-\frac{1}{\sqrt{2\pi k}}\left(\frac{en}{k}\right)^k 2^{1-\binom{k}{2}} = n-\frac{2}{\sqrt{2\pi k}}\left(\frac{en}{k}\right)^k 2^{-\frac{k(k-1)}{2}}$$

$$> n-\left(\frac{en}{k}\right)^k 2^{-\frac{k(k-1)}{2}} \sim n-\left[2^{\frac{k}{2}}(1+o(1))\right]^k 2^{\frac{-k^2+k}{2}} \quad \left(\text{set } n\sim\frac{k}{e}(1+o(1))2^{k/2}\right)$$

$$\sim \frac{k}{e}(1+o(1))2^{k/2}$$

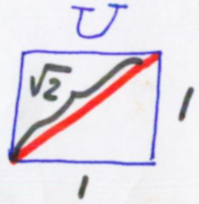
QED

Combinatorial Geometry

- U: a unit square. $T(S) \overset{\text{def}}{=} \min\{ \text{area} \underset{y\quad z}{\overset{x}{\triangle}} : \{x.y.z\} \in \binom{S}{3} \}$. $T(n) \overset{\text{def}}{=} \max_{S \in \binom{U}{n}} T(S)$.

Komlós, Pintz & Szemerédi 1982 : $T(n) = \Omega(\log n / n^2)$

Thm $\exists S \in \binom{U}{n}$ s.t. $T(S) \geq 1/100n^2$.



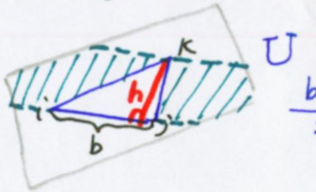
Pf: Select points $p_1, \cdots, p_{2n}$ uniformly and independently from U.

Let $\varepsilon = 1/100n^2$ and $X_{ijk} \overset{\text{def}}{=} \begin{cases} 1 & \text{if area } {}_{p_j}\overset{p_i}{\triangle}{}_{p_k} \leq \varepsilon \\ 0 & \text{o.w.} \end{cases}$

$$\mathcal{E}X_{ijk} = \mathcal{E}(\mathcal{E}(X_{ijk} \mid |p_i - p_j|)) = \int_0^{\sqrt{2}} \underbrace{P(X_{ijk} = 1 \mid |p_i - p_j| = b)}_{\bigstar} f(b)\, db \overset{\triangle}{=}$$ where f is the pdf of $|p_i - p_j|$.

$$\bigstar = P_r\Big({}_i\overset{k}{\underset{b}{\triangle}}{}_j \leq \varepsilon \Big) \leq \frac{(\sqrt{2} \cdot \frac{2\varepsilon}{b}) \times 2}{\text{area}(U)} = \frac{4\sqrt{2}\varepsilon}{b}$$, since

two rectangle



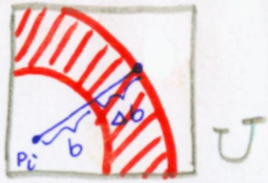
$\frac{b \times h}{2} \leq \varepsilon \Rightarrow h \leq \frac{2\varepsilon}{b}$

To estimate $f(b)$ as follows:

$$f(b) \approx P_r(b \leq |p_i - p_j| \leq b + \Delta b)/\Delta b = \mathcal{E}(P_r(b \leq |p_i - p_j| \leq b + \Delta b \mid p_i))/\Delta b$$

$$= \int_U P_r(b \leq |p_i - p_j| \leq b + \Delta b \mid p_i)\, dP / \Delta b \leq \int_U \frac{\pi(b+\Delta b)^2 - \pi b^2}{\text{area}(U)} dP / \Delta b = 2\pi b + \pi \Delta b \approx 2\pi b$$

$\because P_r(b \leq |p_i - p_j| \leq b + \Delta b) = \int_b^{b+\Delta b} f(t)\,dt \approx f(b)\Delta b$



Thus $\Delta \leq \int_0^{\sqrt{2}} \frac{4\sqrt{2}\varepsilon}{b} \cdot 2\pi b \, db = 16\varepsilon\pi$.

Therefore $\mathcal{E} \sum_{1 \leq i < j < k \leq 2n} X_{ijk} \leq \binom{2n}{3} 16\varepsilon\pi = \frac{2n(2n-1)(2n-2)}{3!} 16 \frac{\pi}{100n^2} < \frac{16\pi 4n^2}{300n} < n$

QED

The Deletion Method

Thm $\alpha(G) \geq \frac{n}{2d}$, provided $v_G = n$ & $e_G = \frac{nd}{2}$, $d \geq 1$.

pf: let S be a random vertex subset with $\Pr(v \in S) = p$ for $\forall v \in V_G$.

Let $X_v = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{o.w.} \end{cases}$, $Y_{xy} = \begin{cases} 1 & \text{if } x \in S \text{ \& } y \in S \\ 0 & \text{o.w.} \end{cases}$

Let $Z = \sum_{v \in V_G} X_v - \sum_{xy \in E_G} Y_{xy}$.

Then $\mathcal{E}Z = np - \frac{nd}{2}p^2$. set $p = \frac{1}{d}$ to maximize $\mathcal{E}Z$. i.e. $\mathcal{E}Z = \frac{n}{2d}$.

QED

Remark: **A formal definition:**

Let $(2^{V_G}, \Pr)$ be a probability space with $\Pr(\omega) = p^{|\omega|}(1-p)^{n-|\omega|}$ for $\forall \omega \in 2^{V_G}$. Define $X_v, Y_{xy} : 2^{V_G} \to \mathbb{R}$ s.t.

$X_v(\omega) = \begin{cases} 1 & \text{if } v \in \omega \\ 0 & \text{o.w.} \end{cases}$ $Y_{xy}(\omega) = \begin{cases} 1 & \text{if } x \in \omega \text{ \& } y \in \omega \\ 0 & \text{o.w.} \end{cases}$

Packing Constant I

- $f(x) = \max\{ |\mathcal{F}| : \mathcal{F}$ is a family of mutually disjoint copies of C, all lying inside $[0.x]^d \}$, where C is a bounded measurable subset of $\mathbb{R}^d$, i.e. $\mu(C) < \infty$.
- $\delta(C) \overset{def}{=} \lim\limits_{x \to \infty} \frac{\mu(C) f(x)}{x^d}$ the maximal proportion of space that maybe be packed by copies of C, (this limit can be proven always to exist!)

Thm Let C be bounded, convex, and centrally symmetric around the origin.

Then $\delta(C) \geq 2^{-d-1}$.

pf: let $P_1, \cdots, P_n$ be selected independently and uniformly from $[0.x]^d$ i.e. density fun $g(x) = \frac{1}{x^d}$.

Let r.v. $X_{ij} = \begin{cases} 1 & \text{if } (C+P_i) \cap (C+P_j) \neq \phi \\ 0 & \text{o.w.} \end{cases}$, where $C+p = \{ c+p : c \in C \}$, and $X \overset{def}{=} \sum_{1 \leq i < j \leq n} X_{ij}$.

note 1

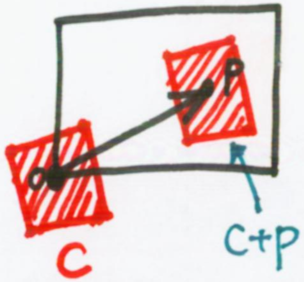
$$\mathcal{E}X = \sum_{1 \leq i < j \leq n} P_r((C+P_i) \cap (C+P_j) \neq \phi)$$

$$\overset{\text{note 1}}{\leq} \sum_{1 \leq i < j \leq n} P_r\{ P_i - P_j \in 2C \} = \sum_{1 \leq i < j \leq n} \mathcal{E} P_r(P_i \in 2C + P_j \mid P_j) = \sum_{1 \leq i < j \leq n} \int_\Omega P_r(P_i \in 2C + P_j \mid P_j) dp$$

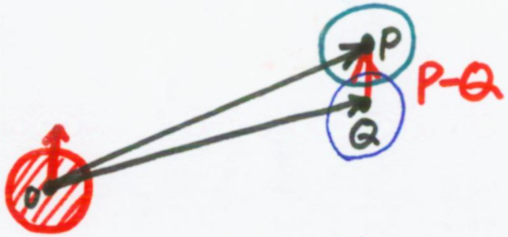
$$\leq \sum_{1 \leq i < j \leq n} \frac{\mu(2C)}{x^d} = \sum_{1 \leq i < j \leq n} \frac{2^d \mu(C)}{x^d} \leq \frac{n^2}{2} \frac{2^d \mu(C)}{x^d}. = D$$

Packing Constant II

note1:

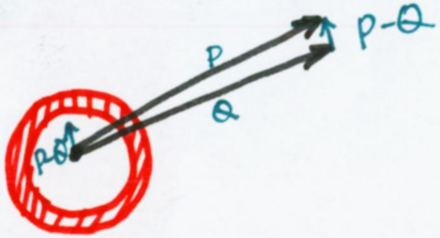


note2

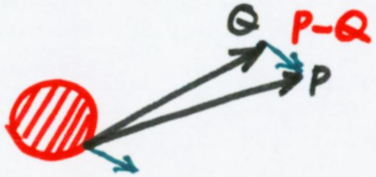


$\Rightarrow \underset{P_i}{\overset{\shortparallel}{P}}-\underset{P_j}{\overset{\shortparallel}{Q}} \in 2C$

note 2' If C is not convex, then we may have $P-Q \notin 2C$



note2'' If C is not centrally symmetric around the origin, then we may have $P-Q \notin 2C$



pf (continued) $\exists$ $\omega_1, \dots, \omega_n \in [0,x]^d$ s.t. $\underbrace{\#\{(i,j) \mid 1 \le i < j \le n,\ (C+\omega_i)\cap(C+\omega_j) \ne \phi\}}_{\le D}$

$\Rightarrow \exists$ at least $n-D$ nonintersecting copies of $C+\omega_k$'s.

$\Rightarrow \exists$ at $(x^d 2^{-d-1})/\mu(C)$ nonintersecting copies of by letting $n = \frac{x^d 2^{-d}}{\mu(C)}$ to minimize $n-D$

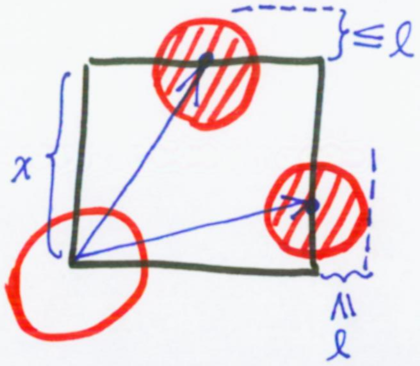
Let $l = \sup\{|x_1|+|x_2|+\cdots+|x_d| : (x_1, \dots, x_d) \in C\}$. Then $f(x+2l) \ge \frac{x^d 2^{-d-1}}{\mu(C)}$. (see note3)

So $\liminf_{n\to\infty} \frac{f(x+2l)\,\mu(C)}{(x+2l)^d} \cdot \frac{(x+2l)^d}{x^d} \ge 2^{-d-1}$. Therefore $\delta(C) \ge 2^{-d-1}$

QED

Packing Constant III

note 3:



and $\mu(C) < \infty \Rightarrow l < \infty$

Greedy Algorithm: Let $p_1, \cdots, p_m$ be any maximal subsets of $[0, x]^d$ with the property that sets $C + p_i$ are disjoint.

Maximality $\Rightarrow$ for any $p \in [0, x]^d$, there exists a p_i with the property

$$(C + p_i) \cap (C + p) \neq \phi.$$

Note that $(C + p_i) \cap (C + p) \neq \phi \Leftrightarrow p - p_i \in 2C \Leftrightarrow p \in 2C + p_i$

Therefore $[0, x]^d \subseteq \cup_{i=1}^{m} (2C + p_i)$

$$\Rightarrow x^d \leq m \mu(2C) = m 2^d \mu(C)$$

$$\Rightarrow f(x + 2l) \underset{\uparrow}{\geq} m \geq \frac{x^d}{2^d \mu(C)} \Rightarrow \liminf_{x \to \infty} \frac{f(x+2l)\mu(C)}{(x+2l)^d} \frac{(x+2l)^d}{x^d} \geq 2^{-d}$$

by the choice of $p_1, \cdots, p_m$

$$\Rightarrow \delta(C) \geq 2^{-d}.$$

QED

High Girth & High Chromatic Number

Thm (Erdös 1959) For $k \geq 2$ & $g \geq 2$, $\exists G$ s.t. $\text{girth}(G) > g$ & $\chi(G) > k$

pf: Let $\omega \in G_{n,p}$.

$X(\omega) \overset{\text{def}}{=\!=}$ the number of cycles in ω with length $\leq g$

$$Y(\omega) \overset{\text{def}}{=\!=} \begin{cases} 1 & \text{if } \alpha(\omega) \geq a \\ 0 & \text{o.w.} \end{cases}$$

$$P\left(X < \tfrac{n}{2} \text{ and } Y < 1\right) \geq 1 - P\left(X \geq \tfrac{n}{2}\right) - P(Y \geq 1)$$

$$\geq 1 - \frac{\mathcal{E}X}{n/2} - \mathcal{E}Y$$

$$\mathcal{E}X \leq \mathcal{E}\left(\sum_{\ell=3}^{g} \sum_{(v_1, \cdots, v_\ell) \in \left[\begin{smallmatrix} n \\ \ell \end{smallmatrix}\right]!} \frac{X_{(v_1, \cdots, v_\ell)}}{2\ell} \right)$$

$$= \sum_{\ell=3}^{g} \binom{n}{\ell} \ell! \frac{1}{2\ell} p^\ell \leq g(np)^g \quad ☆$$

pf (continued)

$$\mathcal{E}Y = P_r\{Y=1\} = P_r\Big\{\bigcup_{S\in[{n \atop a}]}\{Y_S=1\}\Big\}$$

$$\leq \sum_{S\in[{n \atop a}]} P(Y_S=1) = \binom{n}{a}(1-p)^{\binom{a}{2}} \leqslant \Big[e^{\ln n - \frac{p(a-1)}{2}}\Big]^a \bigstar$$

Find p, a so that $\frac{\star}{n/2} \longrightarrow 0$ and $\bigstar \longrightarrow 0$ as $n\to\infty$.

Set $p = n^{\frac{1}{2g}-1}$, $a = \left\lceil \frac{3\ln n}{p} \right\rceil$.

Thus, for n sufficiently large, there exists a graph G of n vertices with $X(G) < \frac{n}{2}$ and $\alpha(G) < a$

Next, delete one vertex from each "small cycle" of G to get G' having

$$\chi(G') \geq \frac{n/2}{\alpha(G')} \geq \frac{n/2}{\alpha(G)} \geq \frac{n/2}{a-1} \geq \frac{n^{\frac{1}{2g}}}{6\ln n} \blacktriangle$$

It remains to choose n so that $\blacktriangle > K$.

QED

Recoloring

- $m(n) = \min\{|\mathcal{E}| : \mathcal{H}=(\mathcal{V}.\overset{\text{multiset}}{\mathcal{E}})$ is an n-uniform hypergraph that is not 2-colorable$\}$

  i.e. ∃ a 2-coloring of V s.t. no edge is monochromati

Thm If $\exists\, p\in[0.1]$ and $k\in\mathbb{N}$ s.t. $k(1-p)^n+k^2p<1$ then $m(n)>2^{n-1}k$.

pf: To show that if n-uniform hypergraph $\mathcal{H}=(\mathcal{V}.\mathcal{E})$ has $|\mathcal{E}|=2^{n-1}k$ then $\mathcal{H}$ is 2-colorable. let $|V|=v$.

a random 2-coloring

random sources: $\Pr(X_v=\overset{\text{red}}{1})=\Pr(X_v=\overset{\text{blue}}{0})=\frac{1}{2}\ \forall v\in\mathcal{V}$. $\mathbf{X}\overset{\text{def}}{=}(X_v)_{v\in\mathcal{V}}$

出現這要變色

$\Pr(Y_v=1)=p=1-\Pr(Y_v=0)$, $\forall v\in\mathcal{V}$. σ is a random permutation of $\mathcal{V}$.

$\Omega=\{(\underset{\sim}{c},\underset{\sim}{b},\sigma):\underset{\sim}{c}.\underset{\sim}{b}\in\{0,1\}^v,\ \sigma\in S_v\}$.

Step I: 根据第一枚銅板 X_v 来 color v.

$$M(\mathbf{X})\overset{\text{def}}{=}\{v\in\mathcal{V}: v \text{ lies in some monochromatic edge under random 2-coloring } \mathbf{X}\}$$

$$=\bigcup_{\substack{e\in\mathcal{E}\\ \text{s.t.}(X_v=1 \text{ for each } v\in e)\\ \text{or } (X_v=0 \text{ for each } v\in e)}} e$$

pf (continuous)

Step 2.1 Run through $M(X)$ sequentially according to σ.

Def: When d is being considered, if ($d \in$ some monochromatic edge e in the first coloring) and (no vertices in e have yet changed color) then d is called **"still dangerous"**

Step 2.2 If d is not still dangerous then do nothing.
else **if** second coin $Y_d = 1$ **then** change the color of d **else** do nothing.

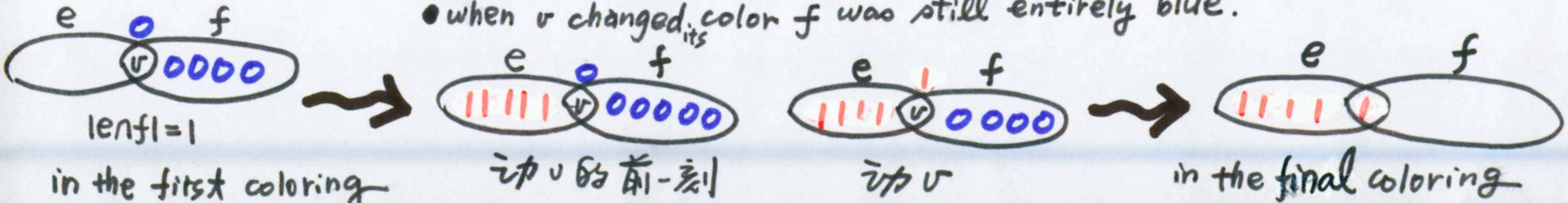
- Pr(some edge is red in the final coloring)

$$= \bigcup_{e \in \xi} Pr(\underbrace{e \text{ is red in the final coloring}}_{\overset{2}{Red}}) \le \sum_{e \in \xi} Pr(\overset{2}{Red}) = \sum_{e \in \xi} Pr(A_e) + \sum_{e \in \xi} Pr(C_e), \text{ where}$$

$$A_e \overset{def}{=} \{\underbrace{e \text{ is red in first coloring}}_{\overset{1}{Red}}\} \cap \overset{2}{Red}, \quad C_e \overset{def}{=} \{\underbrace{e \text{ is not red in the first coloring}}_{\overline{\overset{1}{Red}}}\} \cap \overset{2}{Red}$$

$$Pr(A_e) = Pr(\overset{2}{Red} \mid \overset{1}{Red}) Pr(\overset{1}{Red}) = (1-p)^n \left(\tfrac{1}{2}\right)^n$$

- we say **e blames f** if
  - $|e \cap f| = 1$, say $\{v\} = e \cap f$.
  - In the $\begin{cases}\text{first} \\ \text{final}\end{cases}$ coloring $\begin{cases} f \\ e \end{cases}$ was $\begin{cases}\text{blue.} \\ \text{red.}\end{cases}$
  - v was the last vertex of e that change colors from blue to red
  - when v changed its color f was still entirely blue.



pf (continuous)

claim: $\underbrace{\overline{Red^1}\cap Red^2}_{C_e} \subseteq \overbrace{\{e \text{ blames edge } f\}}^{B_{ef}}$ for some $f\in\xi$.

pf: $\overline{Red^1}\cap Red^2 \Rightarrow$ e 中有一$\overset{v}{\text{点}}$最后变色且由 $\overset{\text{blue}}{0}\rightarrow\overset{\text{red}}{1}$

v 要能夠变色必須滿足 (1) 在 first coloring 中, v 包含在某$\overset{\text{monochromatic edge}}{\text{單色边}}$, say f, 中, 且

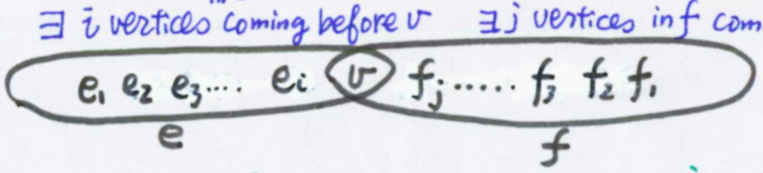
(2) f 中点到目前尚未被改过颜色

∵ condition (1)　　∵ condition (1)+(2)　　∵ v 为 e 中最后一个变色的.

QED of claim

$|e\cap f|=1$ why?

$\Pr(B_{ef}) = \mathcal{E}[\Pr(B_{ef}\mid\sigma)]$ (assume



in the ordering σ.)

see next page $\le \mathcal{E}[2^{1-2n}p(1-p)^j(1+p)^i]$ (note that $i=i(\sigma)$, $j=j(\sigma)$ i.e. i and j are rvs)

$= 2^{1-2n}p\,\mathcal{E}[(1-p)^j(1+p)^i]$

See Claim A next page $\le 2^{1-2n}p$. Therefore we have

$\Pr(\text{the algorithm fails}) \le \Pr(\text{some edge is red in the final coloring}) + \Pr(\text{some edge is blue in the final coloring})$

$\le 2\sum_{e\in\xi}[\Pr(A_e)+\Pr(C_e)] \le 2\sum_{e\in\xi}[(1-p)^n(\frac{1}{2})^n] + 2\sum_{e\in\xi}\Pr(\bigcup_{f\in\xi\setminus e} B_{ef})$

$\le 2(2^{n-1}k)2^{-n}(1-p)^n + 2\sum_{e\neq f}\Pr(B_{ef}) \le k(1-p)^n + 2|\xi|^2 2^{1-2n}p = k(1-p)^n + k^2p$. Done!

Pf (continued) Define the following events

$E_1 = \{\sigma = \sigma^*\}$ where σ^* is a permutation of V.

$E_2 = \{ f$ is blue in the first coloring$\}$

$E_3 = \{Y_{f_1} = Y_{f_2} = \cdots = Y_{f_j} = 0\}$, $\quad E_4 = \{Y_v = 1\}$

$E_5 = \{ X_u = 1$ for each $u \in e \setminus \{e_1, e_2, \dots, e_i, v\} \}$

$E_6 = \{$ for each $u \in \{e_1, e_2, \dots, e_i\}$ either $(X_u \underset{\text{red}}{=1})$ or $(X_u \underset{\text{blue}}{=0}$ & $Y_u = 1) \}$

$$\Pr(\mathrm{Bef} \mid \sigma = \sigma^*) = \Pr(E_2 \mid E_1) \cdots\cdots\cdots \left(\tfrac{1}{2}\right)^n$$

$$\times \Pr(E_3 \mid E_2 \cap E_1) \cdots\cdots\cdots (1-p)^j$$

$$\times \Pr(E_4 \mid E_3 \cap E_2 \cap E_1) \cdots\cdots\cdots p$$

$$\times \Pr(E_5 \mid \cap_{\ell=1}^{4} E_\ell) \cdots\cdots\cdots \left(\tfrac{1}{2}\right)^{n-i-1}$$

$$\times \Pr(E_6 \mid \cap_{\ell=1}^{5} E_\ell) \cdots\cdots\cdots \left(\tfrac{1}{2} + \tfrac{1}{2} \cdot p\right)^i$$

$$\times \Pr(\mathrm{Bef} \mid \cap_{\ell=1}^{6} E_\ell) \cdots\cdots\cdots \leq 1$$

$$\leq \left(\tfrac{1}{2}\right)^n (1-p)^j \, p \left(\tfrac{1}{2}\right)^{n-i-1} \left(\tfrac{1}{2} + \tfrac{p}{2}\right)^i = 2^{1-2n} p (1-p)^j (1+p)^i$$ **QED**