Estimating Binomial Coefficient

Warm up: 1. If $X_1, X_2, \cdots, X_n \xrightarrow{iid} B(1,p)$ Bernoulli r.v.s, then

$\sum_{i=1}^n X_i \sim B(n,p)$ Binomial distribution.

2. If $S \sim B(n,p)$ then $\mathcal{E}S = np$, $Var(S) = npq$.

pf: $P_r(\sum_{i=1}^n X_i = k) = \binom{n}{k} p^k (1-p)^{n-k}$ $k = 0, 1, 2, \ldots, n$.

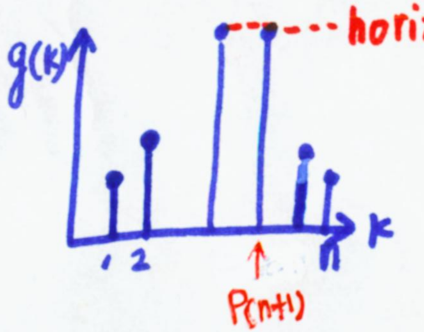
$\mathcal{E}S = \mathcal{E}(\sum_{i=1}^n X_i) = \sum_{i=1}^n p = np$. $Var(S) = Var(\sum_{i=1}^n X_i) = \mathcal{E}(\sum_{i=1}^n (X_i - \mathcal{E}X_i))^2$

$= \sum_{i=1}^n Var X_i + 2\sum_{1 \le i < j \le n} Cov(X_i, X_j) = npq$.

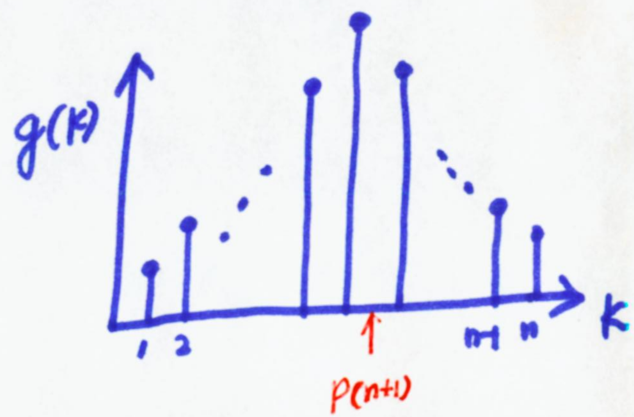
QED

Fact: let $g(k) = \binom{n}{k} p^k (1-p)^{n-k}$. Then we have

$p(n+1) \in \mathbb{Z} \Rightarrow$



$p(n+1) \notin \mathbb{Z} \Rightarrow$



pf: $g(k)/g(k-1) = 1 + \frac{(n+1)p - k}{k(1-p)}$.

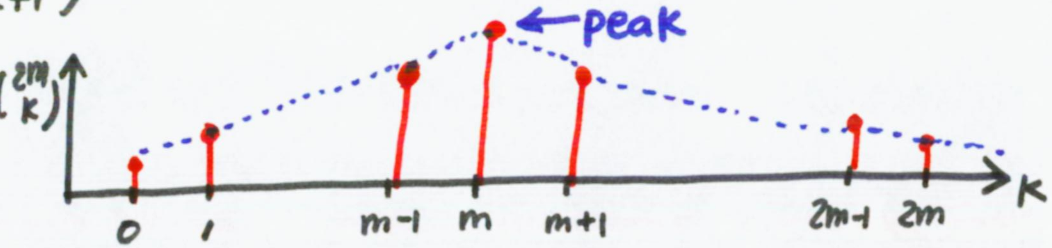
QED

Bounding $\binom{2m}{m}$ from below (by a simple way)

Fact $\binom{2m}{m} = \max_{0 \le k \le 2m} \binom{2m}{k}$.

pf: $\binom{2m}{k} / \binom{2m}{k+1} = (k+1)/(2m-k) \le 1 \iff 2k \le 2m-1$

So we have



QED

Thm For all $m \ge 1$, we have $\binom{2m}{m} \ge 2^{2m}/(4\sqrt{m}+2)$.

note: Stirling's formula says $\binom{2m}{m} \sim 2^{2m}/\sqrt{\pi}\sqrt{m}$

pf: Consider $X \sim B(2m, \frac{1}{2})$

$$P(|X-m| \ge \sqrt{m}) \le \frac{\mathrm{Var}X}{m} = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} \le P(|X-m| < \sqrt{m}) = \sum_{|k|<\sqrt{m}} P(X=m+k) = \sum_{|k|<\sqrt{m}} \binom{2m}{m+k} \left(\frac{1}{2}\right)^{m+k} \left(\frac{1}{2}\right)^{m-k}$$

$$= \left(\frac{1}{2}\right)^{2m} \sum_{|k|<\sqrt{m}} \binom{2m}{m+k} \le \left(\frac{1}{2}\right)^{2m} (2\sqrt{m}+1) \binom{2m}{m} \quad (\because \text{above fact})$$

Therefore $\binom{2m}{m} \ge 2^{2m}/(4\sqrt{m}+2)$

QED

Markov's Inequality

Thm For $t > 0$, $P(|X| \geq t) \leq \frac{E|X|}{t}$.

pf: $E|X| = \int_\Omega |X|\, dP \geq \int_{|X| \geq t} |X|\, dP \geq t P(|X| \geq t)$.

Chebyshev's Inequality

Thm $P(|X - EX| \geq \lambda) \leq \frac{\operatorname{Var} X}{\lambda^2}$ for any $\lambda > 0$.

pf:

$$LHS \leq \frac{E(X - EX)^2}{\lambda^2} = RHS.$$

Cauchy schwarz inequality

Thm $(\Sigma XY)^2 \leq \Sigma X^2 \, \Sigma Y^2.$

Pf: Let $f(\lambda) = \Sigma (X+\lambda Y)^2$

$$= \Sigma X^2 + 2\lambda \Sigma XY + \lambda^2 \Sigma Y^2 \geq 0 \text{ for any } \lambda \in \mathbb{R}.$$

So $(2\Sigma XY)^2 - 4 \;\; \Sigma X^2 \, \Sigma Y^2 \leq 0.$

i.e. $(\Sigma XY)^2 \leq (\Sigma X^2)(\Sigma Y^2).$

QED

Upper bounds for $P(X=0)$

Thm (1) $P(X=0) \leq \dfrac{\operatorname{Var} X}{(\mathcal{E}X)^2}$, provided $\mathcal{E}X \neq 0$.

(2) $P(X=0) \leq \dfrac{\operatorname{Var} X}{\mathcal{E}(X^2)}$, provided $\mathcal{E}X^2 \neq 0$.

pf: (2) $(\mathcal{E}X)^2 = (\mathcal{E}X I_{\{X\neq 0\}})^2$

$$\leq (\mathcal{E}X^2)(\mathcal{E}I_{\{X\neq 0\}}) \quad (\because \text{Cauchy-Schwarz Ineq.})$$

$$= (\mathcal{E}X^2)\ (1 - P(X=0))$$

i.e. $(\mathcal{E}X^2)P(X=0) \leq \operatorname{Var} X$.

QED

Threshold Functions

Def: $r(n)$ is called a threshold fun. for a graph property $\mathcal{A}$ provided

$$\lim_{n\to\infty} \frac{p(n)}{r(n)} = 0 \Longrightarrow \lim_{n\to\infty} \Pr(\mathcal{G}_{n,p(n)} \in \mathcal{A}) = 0$$

Erdös & Rényi
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i.e. $p(n) << r(n)$

$p(n) = o(r(n))$

$$\lim_{n\to\infty} \frac{p(n)}{r(n)} = \infty \Longrightarrow \lim_{n\to\infty} \Pr(\mathcal{G}_{n,p(n)} \in \mathcal{A}) = 1$$

i.e. $p(n) >> r(n)$

$r(n) = o(p(n))$

Note: A threshold fun. may not exist and if it exists, it is not unique. For example the property "$G_{n,p}$ contains a Δ" has threshold fun. $r(n) = \frac{1}{n}$ but $r(n) = \frac{c}{n}$ for any $c > 0$ could serve as well!

tf2

Thm $p(n) \ll \frac{1}{n} \Rightarrow \Pr(\mathcal{G}_{n,p} \text{ contains a } \Delta) \longrightarrow 0$

$p(n) \gg \frac{1}{n} \Rightarrow \Pr(\mathcal{G}_{n,p} \text{ contains a } \Delta) \longrightarrow 1$

↑ monotone graph property

i.e. Suppose $V_G = V_H$, $E_G \subseteq E_H$.
monotone graph property means $G \in \mathcal{A} \Rightarrow H \in \mathcal{A}$

pf: Let $X = \sum_{S \in \binom{[n]}{3}} X_S$

- $P(X \geq 1) \leq \mathcal{E}X = \sum_{S \in \binom{[n]}{3}} P(X_S = 1) = \binom{n}{3} p^3 \leq \frac{(np)^3}{6} \rightarrow 0$ as $p(n) \ll \frac{1}{n}$.

- $P(X=0) \leq P(|X - \mathcal{E}X| \geq \mathcal{E}X) \leq \frac{\operatorname{Var} X}{(\mathcal{E}X)^2}$

$$= \left(\sum_{S \in \binom{[n]}{3}} \operatorname{Var} X_S + \sum_{|S \cap S'| = 2} \operatorname{Cov}(X_S, X_{S'}) \right) \Big/ (\mathcal{E}X)^2$$

$$\leq \sum_{S} \mathcal{E} X_S^2 + \sum_{|S \cap S'| = 2} \mathcal{E}(X_S X_{S'}) \Big/ (\mathcal{E}X)^2$$

$$= \left[\binom{n}{3} p^3 + \binom{n}{4}\binom{4}{2} 2 \; p^5 \right] \Big/ \left(\binom{n}{3} p^3 \right)^2$$

$$= \frac{1}{\binom{n}{3} p^3} + \frac{12 \binom{n}{4} p^5}{\binom{n}{3}^2 p^6} \longrightarrow 0 \text{ as } p(n) \gg \frac{1}{n}$$

QED

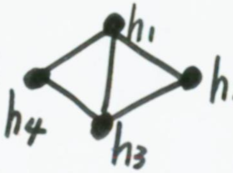
Balanced Graphs: special case

Def: A graph is balanced if its average degree is at least as large the average degree of any its induced subgraphs.

Ex: Trees, cycles, connected unicyclic graphs, complete graphs....

Thm $p \ll n^{-\frac{4}{5}} \Longrightarrow \Pr\{$  $\overset{\text{subgraph}}{\subseteq} G_{n,p}\} \longrightarrow 0$ as $n \to \infty$.

$p \gg n^{-\frac{4}{5}} \Longrightarrow \Pr\{$  $\subseteq G_{n,p}\} \longrightarrow 1$ as $n \to \infty$.

pf: $V_{G_{n,p}} \equiv [n]$, $H \equiv$ 

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For $s \in \binom{[n]}{4}!$, $s = (s_1, s_2, s_3, s_4)$.

Let $X_s^{(\omega)} = \begin{cases} 1 & \text{if } h_i \to s_i \ \ 1 \le i \le 4 \text{ is a homomorphism from } H \text{ to } \omega. \\ 0 & \text{o.w.} \end{cases}$

pf (continued) Let $X = \sum_{s \in \binom{[n]}{4}!} X_s$.

- $\Pr\{$  $\subseteq G_{n,p}\} \le \Pr\{X \ge 1\} \le \mathcal{E}X = \sum_{s \in \binom{[n]}{4}!} \mathcal{E}X_s = 4!\binom{n}{4}p^5$

$= (n^{\frac{4}{5}} p)^5 \longrightarrow 0$ as $p << n^{-\frac{4}{5}}$.

- $\Pr\{$  $\subseteq G_{n,p}\} = 1 - \Pr\{X = 0\}$.

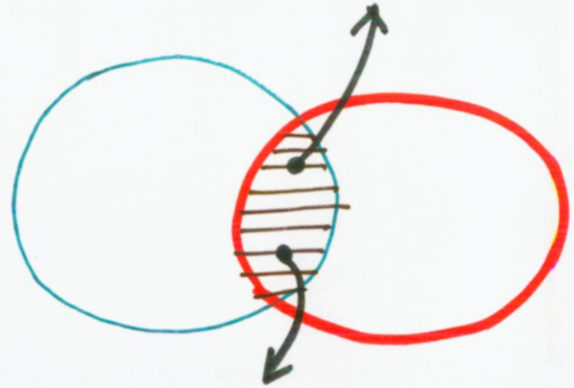
$$\Pr\{X=0\} \le \Pr\{|X - \mathcal{E}X| \ge \mathcal{E}X\} \le \frac{\operatorname{Var}X}{(\mathcal{E}X)^2} \;\star$$

$$= \frac{\sum_s \operatorname{Var}X_s + \sum_{|s \cap s'| \ge 2,\ s \ne s'} \operatorname{Cov}(X_s, X_{s'})}{(\mathcal{E}X)^2}$$

$$\sum_s \operatorname{Var}X_s \le \sum_s \mathcal{E}X_s^2 = \sum_s \mathcal{E}X_s$$
$$= \mathcal{E}X$$

pf (continued)

vertex # $= t$



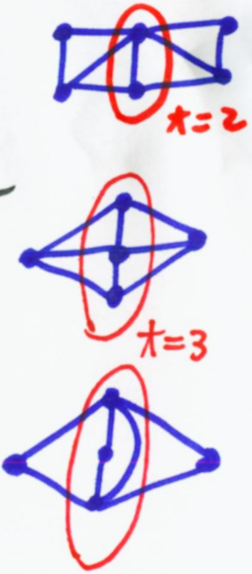
edge # $\leq \frac{5}{4}t$

$$\star \leq \sum_{t=2}^{3} \sum_{|s\cap s'|=t} \mathcal{E}(X_s X_{s'})$$

$$\leq \sum_{t=2}^{3} \sum_{|s\cap s'|=t} p^{2\cdot 5 - \frac{5}{4}t}$$

$$\leq \sum_{t=2}^{3} O\left(\binom{n}{2\cdot 4 - t}\right) p^{10-\frac{5}{4}t}$$

$$\leq \sum_{t=2}^{3} O(n^{8-t})\, p^{10-\frac{5}{4}t} \equiv \Delta.$$



So $\Pr\{X=0\} \leq \dfrac{1}{\mathcal{E}X} + \dfrac{\Delta}{[(n-3)^4 p^5]^2}$, where

$$\frac{1}{\mathcal{E}X} \leq \frac{1}{(n-3)^4 p^5} \sim \frac{1}{(n^{\frac{4}{5}}p)^5} \longrightarrow 0 \text{ as } p \gg n^{-\frac{4}{5}}$$

$$\frac{\Delta}{[(n-3)^4 p^5]^2} \sim \frac{\Delta}{n^8 p^{10}} = \sum_{t=2}^{3} O(n^{-t} p^{-\frac{5}{4}t})$$

$$= \sum_{t=2}^{3} O\left((n^{\frac{4}{5}}p)^{-\frac{5}{4}t}\right) \longrightarrow 0 \text{ as } p \gg n^{-\frac{4}{5}}$$

QED

Balanced Graphs : general case

Def: G is balanced if any induced subgraph H of G has $\frac{e_G}{v_G} \geq \frac{e_H}{v_H}$.

Thm Let H be a balanced graph with v vertices and e edges.

then $p(n) \ll n^{-\frac{v}{e}} \Rightarrow \Pr(H \underset{\text{subgraph}}{\subseteq} \mathcal{G}_{n,p}) \longrightarrow 0$ as $n \to \infty$

$p(n) \gg n^{-\frac{v}{e}} \Rightarrow \Pr(H \subseteq \mathcal{G}_{n,p}) \longrightarrow 1$ as $n \to \infty$

Pf: let $X = \sum_{\underset{\sim}{s} \in \binom{[n]}{v}!} X_{\underset{\sim}{s}}$

- $\Pr(X \geq 1) \leq \sum_{\underset{\sim}{s} \in \binom{[n]}{v}!} \mathcal{E} X_{\underset{\sim}{s}} = v! \binom{n}{v} p^e \leq n^v p^e = (n^{\frac{v}{e}} p)^e \to 0$ as $p(n) \ll n^{-\frac{v}{e}}$

- $\Pr(X = 0) \leq \Pr(|X - \mathcal{E}X| \geq \mathcal{E}X) \leq \frac{\operatorname{Var} X}{(\mathcal{E}X)^2}$

$$\leq \Bigg(\underbrace{\sum_{\underset{\sim}{s}} \mathcal{E} X_{\underset{\sim}{s}}^2}_{1} + \underbrace{\sum_{\substack{|\underset{\sim}{s} \cap \underset{\sim}{s}'| \geq 2 \\ \underset{\sim}{s} \neq \underset{\sim}{s}'}} \mathcal{E}(X_{\underset{\sim}{s}} X_{\underset{\sim}{s}'})}_{2} \Bigg) \Big/ (\mathcal{E}X)^2$$

pf (continued)

$$① = \nu! \binom{n}{\nu} p^e$$

$$② \le \sum_{t=2}^{\nu-1} \sum_{|\underset{\sim}{s} \cap \underset{\sim}{s}'| = t} p^{2e - t\frac{e}{\nu}} \quad (\because H \text{ is a balanced graph})$$

$$= \sum_{t=2}^{\nu-1} O\left(\binom{n}{2\nu - t}\right) p^{2e - t\frac{e}{\nu}} \quad (\because \nu, e \text{ are constant})$$

Thus
$$\Pr(X=0) \le \frac{①}{(\mathcal{E}X)^2} + \frac{②}{(\mathcal{E}X)^2}$$

$$= \frac{1}{\Theta(n^\nu p^e)} + \frac{\sum_{t=2}^{\nu-1} O(n^{2\nu - t}) p^{2e - t\frac{e}{\nu}}}{[\Theta(n^\nu p^e)]^2}$$

$$= \frac{1}{\Theta(n^\nu p^e)} + \frac{\sum_{t=2}^{\nu-1} O\left((n^\nu p^e)^{2 - \frac{t}{\nu}}\right)}{[\Theta(n^\nu p^e)]^2}$$

$$= \frac{1}{\Theta(n^\nu p^e)} + \sum_{t=2}^{\nu-1} O\left((n^\nu p^e)^{-\frac{t}{\nu}}\right) \longrightarrow 0$$

as $p(n) \gg n^{-\frac{\nu}{e}}$

QED

Threshold Function: isolated vertices

Thm

$p(n) \gg \frac{\ln n}{n} \Longrightarrow \Pr(G_{n,p}$ contains isolated vertices$) \to 0$

$p(n) \ll \frac{\ln n}{n} \Longrightarrow \Pr(G_{n,p}$ contains isolated vertices$) \to 1$

pf:

$$X_v = \begin{cases} 1 & v \text{ is an isolated vertex in } G. \\ 0 & \text{o.w.} \end{cases}$$

- $\Pr(G_{n,p}$ contains an isolated vertex$) = P\left(\sum_{v \in V} X_v \geq 1\right) \leq \sum_{v \in V} \mathcal{E} X_v$

$$= n(1-p)^{n-1} \leq n e^{-p(n-1)} = e^{\ln n - p(n-1)} \to 0 \text{ as } p(n) \gg \frac{\ln n}{n}.$$

- $\Pr(G_{n,p}$ contains **no** isolated vertex$) = P\left(\sum_v X_v = 0\right)$

$$\leq P(|X - \mathcal{E}X| \geq \mathcal{E}X) \leq \frac{\operatorname{Var} X}{(\mathcal{E}X)^2} \text{ where } X = \sum_v X_v.$$

pf (continued)

$$\mathrm{Var}X = \sum_{i=1}^{n} \mathrm{Var}X_i + 2\sum_{1\le i<j\le n} \mathrm{cov}(X_i, X_j)$$

$$\le \sum_{i=1}^{n} \mathcal{E}X_i^2 + 2\sum_{1\le i<j\le n}\left(\mathcal{E}X_iX_j - \mathcal{E}X_i\,\mathcal{E}X_j\right)$$

$$= n(1-p)^{n-1} + 2\binom{n}{2}\left((1-p)^{2(n-2)+1} - (1-p)^{2(n-1)}\right)$$

$$(\mathcal{E}X)^2 = n^2(1-p)^{2(n-1)}$$

Note that $\dfrac{2\binom{n}{2}(1-p)^{2n-3}}{n^2(1-p)^{2n-2}} \sim (1-p)^{-1} \to 1$ as $n\to\infty$ under $p(n) \ll \dfrac{\ln n}{n}$

$$\frac{2\binom{n}{2}(1-p)^{2n-2}}{n^2(1-p)^{2n-2}} \to 1 \text{ as } n\to\infty$$

$$\frac{n(1-p)^{n-1}}{n^2(1-p)^{2n-2}} = \frac{1}{n(1-p)^{n-1}}$$

$$\sim \frac{1}{n e^{-p(n-1)}} \quad (\because p(n)\to 0 \text{ as } n\to\infty)$$

$$= e^{-\ln n + p(n-1)} = e^{\ln n\left(\frac{p}{\frac{\ln n}{n-1}} - 1\right)} \to 0 \text{ as } n\to\infty \text{ under } p(n) \ll \frac{\ln n}{n}$$

Thus $\dfrac{\mathrm{Var}X}{(\mathcal{E}X)^2} \to 0$ as $p(n) \ll \dfrac{\ln n}{n}$.

QED

Clique Number (1)

Fact: For any fixed $r \in \{2, 3, 4, \cdots\}$ and fixed $p \in [0, 1]$, almost all graphs $G_{n,p}$ contains a complete subgraph K_r

i.e. $\Pr(K_r \subseteq G_{n,p}) \to 1$ as $n \to \infty$.

pf: K_r is a balanced graph with $n^{-\frac{v}{e}} = n^{-\frac{r}{\binom{r}{2}}}$.

$$p / n^{-\frac{v}{e}} = p n^{\frac{2}{r-1}} \to \infty \text{ as } n \to \infty \Rightarrow p >> n^{-\frac{v}{e}}$$

$$\Rightarrow \Pr(K_r \subseteq G_{n,p}) \to 1 \text{ as } n \to \infty.$$

($\because$ the threshold funs. for K_r)

QED

Clique number (2)

Question Let X_r = the # of r-cliques in $G_{n,\frac{1}{2}}$. We want to seek a solowly increasing function $r = r(n)$ s.t. $\mathcal{E}X_r \to 0$ as $n \to \infty$.

Solution:

Let $X_r \overset{def}{=\!=} \sum_{S \in [n]_r} X_S$, where $X_S \overset{def}{=\!=} \begin{cases} 1 & \text{if } \omega[S] \cong K_r \\ 0 & \text{o.w.} \end{cases}$

$$\mathcal{E}X_r = \binom{n}{r}\left(\frac{1}{2}\right)^{\binom{r}{2}} \leq \left(\frac{ne}{r}\right)^r \left(\frac{1}{2}\right)^{\frac{r(r-1)}{2}}$$

$$= \left(\frac{ne}{r}\left(\frac{1}{2}\right)^{\frac{r-1}{2}}\right)^r \overset{def}{=\!=} \star(n).$$

Note that $\left(\lim_{n\to\infty} r(n) = \infty\right) + \left(\frac{ne}{r}\left(\frac{1}{2}\right)^{\frac{r-1}{2}} < 1\right) \Rightarrow \lim_{n\to\infty} \star(n) = 0.$

And $\left(\frac{ne}{r}\right)\left(\frac{1}{2}\right)^{\frac{r-1}{2}} < 1 \Longleftrightarrow r > 2\log_2 n - 2\log_2 r + 2\log_2 e - 1$

Therefore $r > 2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 1 \Rightarrow \lim\limits_{n\to\infty} \mathcal{E}X_r = 0.$

END

Concentration result for clique number

Thm (Matula 1972, Bollobás & Erdős 1976) Given $\varepsilon > 0$. There exists $k = k(n)$ so that

$$\Pr\{\omega(G_{n,\frac{1}{2}}) \in [\lfloor k-\varepsilon \rfloor, \lfloor k+\varepsilon \rfloor]\} \to 1.$$

pf: Let $k = 2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 1$.

To show $\lim_{n\to\infty} \Pr\{\omega(G_{n,\frac{1}{2}}) > \lfloor k+\varepsilon \rfloor\} = 0$.

let $r = \lceil k+\varepsilon \rceil$, $X_S \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } \omega[S] \cong K_r \\ 0 & \text{o.w.} \end{cases}$ and $X_r \stackrel{\text{def}}{=} \sum_{S \in \binom{[n]}{r}} X_S$

$\Pr\{\omega(G_{n,\frac{1}{2}}) > \lfloor k+\varepsilon \rfloor\} \le \Pr\{\omega(G_{n,\frac{1}{2}}) \ge \lfloor k+\varepsilon \rfloor + 1\}$

$\le \Pr\{\omega(G_{n,\frac{1}{2}}) \ge \lceil k+\varepsilon \rceil\} \le \Pr\{X_r \ge 1\} \le \mathbb{E}X_r \to 0$ as $n \to \infty$,

since $r = r(n) \to \infty$ as $n \to \infty$, and $r = \lceil k+\varepsilon \rceil > k$ (see the discussion in the previous page)

pf (continued) To show $\Pr\{\omega(G_{n,\frac{1}{2}}) \ge \lfloor k-\varepsilon \rfloor\} \longrightarrow 1$.

Let $r = \lfloor k-\varepsilon \rfloor$. $\Pr\{\omega(G_{n,\frac{1}{2}}) \ge r\} = 1 - \Pr\{X_r = 0\}$.

$$\Pr\{X_r=0\} \le \Pr\{|X_r - \mathcal{E}X_r| \ge \mathcal{E}X_r\} \le \mathrm{Var}X_r / (\mathcal{E}X_r)^2$$

$$= \Big[\sum_{S\in\binom{[n]}{r}} \mathrm{Var}X_S + \sum_{\substack{S,T\in\binom{[n]}{r}\\ S\ne T}} \mathrm{Cov}(X_S,X_T)\Big] / (\mathcal{E}X_r)^2 \le \underbrace{\frac{\sum_{S\in\binom{[n]}{r}} \mathcal{E}X_S^2}{(\mathcal{E}X_r)^2}}_{\textcolor{blue}{\bigstar}} + \underbrace{\frac{\sum_{\substack{S,T\in\binom{[n]}{r}\\ |S\cap T|\ge 2,\ S\ne T}} \mathrm{Cov}(X_S,X_T)}{(\mathcal{E}X_r)^2}}_{\textcolor{red}{\bigstar}}.$$

$$\textcolor{blue}{\bigstar} = 1/(\mathcal{E}X_r) = 1/\binom{n}{r}2^{-\binom{r}{2}} \le \frac{r!\,2^{\binom{r}{2}}}{(n-r+1)^r}$$

$$\le \sqrt{2\pi r}\left(\frac{r}{e}\right)^r e^{\frac{1}{12r}} \frac{2^{\frac{r(r-1)}{2}}}{n^r}\left(\frac{n}{n-r+1}\right)^r \le \sqrt{2\pi r}\Big(\underbrace{\frac{r\,2^{\frac{r-1}{2}}}{en}}_{\textcolor{red}{\star}}\Big)^r e^{\frac{1}{12r}} e^{\frac{r(r-1)}{n-r+1}} \quad \left(\because \frac{n}{n-r+1} = 1 + \frac{r-1}{n-r+1}\right)$$

$$\textcolor{red}{\star} \le \frac{r\,2^{\frac{k-\varepsilon-1}{2}}}{en} = \frac{r\,2^{\log_2 n - \log_2\log_2 n + \log_2 e - 1}}{en}\, 2^{-\frac{\varepsilon}{2}} = \frac{r}{2\log_2 n}\, 2^{-\frac{\varepsilon}{2}} \le 2^{-\frac{\varepsilon}{2}}.$$

$(\because r = \lfloor k-\varepsilon \rfloor \le k-\varepsilon)$

$$\textcolor{blue}{\bigstar} \le \sqrt{2\pi r}\,(2^{-\frac{r\varepsilon}{2}})\, \underbrace{e^{\frac{r(r-1)}{n-r+1}} e^{\frac{1}{12r}}}_{\square} = \sqrt{2\pi}\left(\frac{r}{2^{r\varepsilon}}\right)^{\frac{1}{2}} \square \longrightarrow 0 \quad \left(\because \left(\frac{r}{2^{r\varepsilon}}\right) \to 0,\ \square \to 1 \text{ as } n\to\infty\right).$$

$$\textcolor{red}{\bigstar} \le \sum_{T\in\binom{[n]}{r}} \Big[\sum_{S\in\binom{[n]}{r},\ 2\le|S\cap T|\le r-1} P(X_S=1 \mid X_T=1)\Big] P(X_T=1) \Big/ (\mathcal{E}X_r)^2.$$

$$= \underbrace{\Big[\sum_{i=2}^{r-1} \binom{r}{i}\binom{n-r}{r-i}\left(\frac{1}{2}\right)^{\binom{r}{2}-\binom{i}{2}}\Big]}_{\blacksquare} \sum_{T\in\binom{[n]}{r}} P(X_T=1) \Big/ (\mathcal{E}X_r)^2.$$

$$= \blacksquare\, \mathcal{E}X_r / (\mathcal{E}X_r)^2 = \blacksquare \Big/ \binom{n}{r}\left(\frac{1}{2}\right)^{\binom{r}{2}} = \sum_{i=2}^{r-1} \frac{\binom{r}{i}\binom{n-r}{r-i}}{\binom{n}{r}}\, 2^{\binom{i}{2}} \xrightarrow{\text{see Palmer p78}} 0, \text{ as } n\to\infty.$$

QED