Distinct Sums

Fact If $f(n) \overset{\text{def}}{=} \max\{|\mathcal{F}| : \mathcal{F} \subseteq [n], \text{ and } \sum_{a\in S} a \neq \sum_{b \in T} b \text{ for any two distinct subset } S, T \text{ of } \mathcal{F}\}$

we say that $\mathcal{F}$ has distinct sum.

then $1+\lfloor \log_2 n \rfloor \overset{①}{\leq} f(n) \overset{②}{\leq} \log_2 n + \log_2\log_2 n + O(1)$.

Pf: "$\overset{①}{\leq}$" let $\mathcal{F} = \{2^i : i \leq \log_2 n\}$. Then $\mathcal{F}$ has distinct sums. (∵ 2 進位)

"$\overset{②}{\leq}$" $\exists$ a set $\{x_1, \dots, x_{f(n)}\} \subseteq [n]$ having distinct sums.

$$\Rightarrow 2^{f(n)} = \left|\left\{\sum_{a\in S} a : S \subseteq \{x_1, \dots, x_{f(n)}\}\right\}\right| \leq n+(n-1)+\cdots+(n-f(n)+1) < nf(n) \quad \left(\text{here } \sum_{a\in\emptyset} a = 0\right)$$

$$\Rightarrow f(n) < \log_2 n + \log_2 f(n) \Rightarrow f(n) \leq \log_2 n + \log_2\log_2 n + 1 \quad (\because f(n) < 2\log_2 n \text{ implies } \log_2 f(n) < 1 + \log_2\log_2 n)$$

QED

Thm $f(n) \leq \log_2 n + \frac{1}{2}\log_2\log_2 n + O(1)$

Pf: Suppose $\{x_1, \dots, x_k\} \subseteq [n]$ has distinct sums. let $\varepsilon_1, \varepsilon_2, \dots \varepsilon_k \overset{\text{iid}}{\sim} B(1, \frac{1}{2})$.

Consider random sum $X = \sum_{i=1}^k \varepsilon_i x_i$. Using the property that $\{x_1, \dots, x_k\}$ has distinct sum

we have
$$\frac{2t+1}{2^k} \geq P\{|X-EX| \leq t\} = 1 - \Pr(|X-EX| > t) \geq 1 - \frac{\operatorname{Var}X}{t^2} = 1 - \frac{\sum_{i=1}^k x_i^2 \frac{1}{4}}{t^2} \geq 1 - \frac{kn^2}{4t^2}.$$

let $t = \lambda n\sqrt{k}/2$, $\lambda > 1$ ($\because$ choose t so that $1 - \frac{kn^2}{4t^2} > 0$ i.e. $t > n\sqrt{k}/2$)

Therefore $\frac{\lambda n\sqrt{k}+1}{2^k} \geq 1 - \frac{1}{\lambda^2}$ and hence $n \geq \frac{2^k(1-\frac{1}{\lambda^2})-1}{\lambda\sqrt{k}}$. Set $\lambda = 2$ to get

$\log_2 n + \frac{1}{2}\log_2 k + 1 \geq \log_2[2^k \frac{3}{4} - 1] \geq \log_2 2^{k-1} = k-1$. Therefore $k \leq \log_2 n + \frac{1}{2}\log_2 k + 2$ and hence $k \leq \frac{3}{2}\log_2 n + 2$

so $k \leq \log_2 n + \frac{1}{2}\log_2\log_2 n + O(1)$

QED

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Erdös offored \$300 for a proof or disproof that

$$f(n) \leq \log_2 n + C \text{ for some constant } C.$$