Lovász Local Lemma (I)

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Dependency Digraph

Def: A dependency digraph for a set of events $A_1, \cdots, A_n$ is a digraph $\vec{G} = (V, \vec{E})$ s.t. $V = \{1, 2, \cdots, n\}$ and, for each i, $1 \le i \le n$ the event A_i is mutually independent of the events $\{A_j : \vec{ij} \notin \vec{E}\}$.

Note: 1. $E_1, \ldots, E_n$ (events) are mutually independent iff for any subset $I \subseteq [n]$

$$Pr\left(\bigcap_{i \in I} E_i\right) = \prod_{i \in I} Pr(E_i)$$

2.* We say that an event E is mutually independent of the events $E_1, \cdots, E_n$ if for any subset $I \subseteq [n]$, $Pr\left(E \mid \bigcap_{j \in I} \overline{E_j}\right) = Pr(E)$.

Lovász Local Lemma: general case

Thm (LLL, General Case)

Let $D=(V,\vec{E})$ be a dependency digraph of events $A_1, A_2, \cdots, A_n$.

If $\exists\ 0 \le x_i < 1$, s.t. $P(A_i) \le x_i \prod_{j \in N^+(i)} (1-x_j)$, $\forall i$

then $P\left(\bigcap_{i=1}^{n} \overline{A_i}\right) \ge \prod_{i=1}^{n} (1-x_i) > 0$

Pf: **Claim** For $i \notin S \subseteq \{1,2,\cdots,n\}$, $P\left(A_i \mid \bigcap_{j \in S} \overline{A_j}\right) \le x_i$

Pf: By induction on $|S|$.

Let $B = \bigcap_{j \in S_1} \overline{A_j}$, $C = \bigcap_{j \in S_2} \overline{A_j}$ where $S_1 = S \cap N^+(i) \overset{\text{def}}{=} \{s_1, \cdots, s_r\}$. $S_2 = S \setminus S_1$

$P(B \mid C) = P(\overline{A}_{s_1} \mid \overline{A}_{s_2} \cap \cdots \cap \overline{A}_{s_r} \cap C)$

$P(\overline{A}_{s_2} \mid \overline{A}_{s_3} \cap \cdots \cap \overline{A}_{s_r} \cap C)$

$\vdots$

$P(\overline{A}_{s_{r-1}} \mid \overline{A}_{s_r} \cap C)$

$P(\overline{A}_{s_r} \mid C)$

$\ge (1-x_{s_1})(1-x_{s_2}) \cdots (1-x_{s_r})$ by induction hypothesis

$$P(A_i \mid \bigcap_{j\in S} \bar{A}_j) = P(A_i \mid B \cap C)$$

$$= \frac{P(A_i \cap B \cap C)}{P(B \cap C)}$$

$$= \frac{P(A_i \cap B \mid C)}{P(B \mid C)}$$

$$\leq \frac{P(A_i \mid C)}{P(B \mid C)} \overset{*}{=} \frac{P(A_i)}{P(B \mid C)} \leq \frac{x_i \prod_{j\in N^+(i)} (1-x_j)}{(1-x_{s_1})\cdots(1-x_{s_r})} \leq x_i$$

QED of Claim

Therefore $P(\bigcap_{i=1}^{n} \bar{A}_i)$

$$= P(\bar{A}_1 \mid \bigcap_{i=2}^{n} \bar{A}_i)$$

$$P(\bar{A}_2 \mid \bigcap_{i=3}^{n} \bar{A}_i)$$

$$P(\bar{A}_3 \mid \bigcap_{i=4}^{n} \bar{A}_i)$$

$$\vdots$$

$$P(\bar{A}_{n-1} \mid \bigcap_{i=n}^{n} \bar{A}_i)\ P(\bar{A}_n) \geq \prod_{i=1}^{n} (1-x_i) > 0$$

QED

Lovász Local Lemma: symmetric case

Thm (LLL, Symmetric case)

Let D be a dependency digraph of events $A_1, \dots, A_n$ with $d^+(i) \le d \ \forall i$.

If $P(A_i) \le \frac{1}{e(d+1)} \ \forall i$, then $P(\bigcap_{i=1}^{n} \bar{A}_i) > 0$

1/(4d)

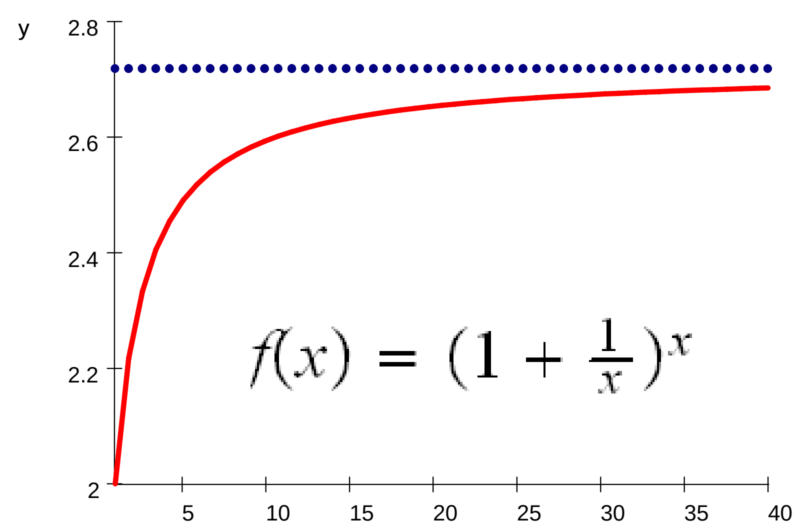
pf: For $d \ne 0$, set $x_i = \frac{1}{d+1} < 1$

$$P(A_i) \le \frac{1}{e(d+1)}$$

$$\le \frac{1}{(1+\frac{1}{d})^d (d+1)}$$

$$= \frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^d$$

$$\le x_i \prod_{j \in N^+(i)} (1 - x_j)$$



QED

Ramsey Numbers

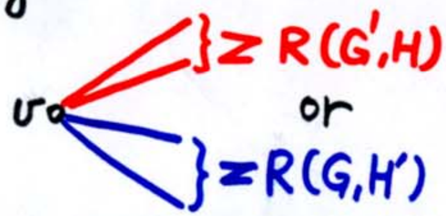
$R(G,H) \overset{\text{def}}{=} \min\{ n : $ every red-blue edge coloring of K_n contains a red G or a blue $H \}$

$R(k,\ell) \overset{\text{def}}{=} R(K_k, K_\ell)$ Facts 1. $r(K_1, G) = 1$ 2. $r(K_2, G) = |V(G)|$.

Thm $\nu_G, \nu_H \geq 2$. Then $R(G,H) \leq R(G',H) + R(G,H')$ (G′ obtained from G by deleting one vertex). Moreover if both $R(G',H)$ & $R(G,H')$ are even then $R(G,H) < R(G',H) + R(G,H')$.

Pf: let $n = R(G',H) + R(G,H')$.

(1) red-blue edge color K_n to get



Thus $R(G,H) \leq n$.

(2) Assume $R(G,H) = n$. So $\exists$ a red-blue edge colo[ring] of K_{n-1} s.t. no red G & no blue H. For every vertex v in K_{n-1}

$< R(G',H)$ and $< R(G,H')$. Thus $= R(G',H) - 1$, $= R(G,H') - 1$.

And $|\text{red edges}| = \dfrac{(n-1)[R(G',H)-1]}{2}$ a contradiction since n and $R(G',H)$ are even.

QED

Bounding R(k,k) with LLL

Thm If $e\left\{\binom{k}{2}\binom{n}{k-2}+1\right\}2\left(\frac{1}{2}\right)^{\binom{k}{2}} \leq 1$ then $R(k,k)>n$

pf:

For $S\in\binom{[n]}{k}$, let $A_S=\{G_{n,\frac{1}{2}}[S]\cong K_k \text{ or } \overline{K_k}\}$.

Note that $\Pr(A_S)\leq 2\Pr(G_{n,\frac{1}{2}}[S]\cong K_k) = 2\left(\frac{1}{2}\right)^{\binom{k}{2}}$,

and $\#\left\{T\in\binom{[n]}{k} : |T\cap S|\geq 2\right\}\leq \binom{k}{2}\binom{n}{k-2}$.

$$\Pr(A_S)\,e\{\Delta+1\}\leq 1$$

Δ: maximum outdegree of the dependency digraph on events A_S.

$$\overset{\text{LLL}}{\Longrightarrow} \Pr\left(\bigcap_{S\in\binom{[n]}{k}} \overline{A}_S\right)>0$$

$$\Longrightarrow R(k,k)>n$$

QED

Corollary $R(k,k) > \frac{\sqrt{2}}{e} k 2^{\frac{k}{2}} (1+o(1))$

Pf: $e\left\{ \binom{k}{2}\binom{n}{k-2} + 1 \right\} 2\left(\frac{1}{2}\right)^{\binom{k}{2}} \leq 1$

$\Leftarrow 3\binom{k}{2}\binom{n}{k-2} 2^{1-\binom{k}{2}} \leq 1$ as $n \to \infty$

$\Leftarrow 3\binom{k}{2} \frac{k(k-1)}{(n-k+2)(n-k+1)} \binom{n}{k} 2^{1-\binom{k}{2}} \leq 1$

$\Leftarrow 3 \frac{k^4}{(n-k)^2} \frac{n^k}{\sqrt{2\pi k}\left(\frac{k}{e}\right)^k} 2^{-\binom{k}{2}} \leq 1$

$\Leftarrow 3 \frac{k^4}{(n-k)^2} \left(\frac{ne}{k}\right)^k 2^{-\frac{k(k-1)}{2}} \leq 1$

$\Leftarrow 3 \frac{k^4}{(n-k)^2} \left\{ \left(\frac{ne}{k}\right) 2^{-\frac{k-1}{2}} \right\}^k \leq 1$

$\Leftarrow 3 \frac{k^4}{k^2\left[\frac{\sqrt{2}}{e} 2^{\frac{k}{2}}(1-\varepsilon) - 1\right]^2} 2^k (1-\varepsilon)^k \leq 1$, here $n \sim \frac{\sqrt{2}}{e} k 2^{\frac{k}{2}} (1-\varepsilon)$

$\Leftarrow 3 \frac{k^2}{\frac{2}{e^2} 2^k (1-\varepsilon)^2} 2^k (1-\varepsilon)^k < 1$, as $k \to \infty$

$\Leftarrow \frac{3e^2}{2} k^2 (1-\varepsilon)^{k-2} < 1$ as $k \to \infty$

QED

Bounding $R(k,\ell)$ with alternation method

Thm For $n \in \mathbb{Z}^+$ and $p \in [0,1]$, $R(k,\ell) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{\ell}(1-p)^{\binom{\ell}{2}}$

Pf: To show $\exists$ a graph of order n with at most $\binom{n}{k} p^{\binom{k}{2}}$ induced K_k and $\binom{n}{\ell}(1-p)^{\binom{\ell}{2}}$ induced $\overline{K_\ell}$. Deleting a single vertex in K_k (resp. $\overline{K_\ell}$) will destroy K_k (resp. $\overline{K_\ell}$). Consider $\mathcal{G}_{n,p}$. $X_R(\omega) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } \omega[R] \cong K_k \\ 0 & \text{o.w.} \end{cases}$; $Y_B(\omega) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } \omega[B] \cong \overline{K_\ell} \\ 0 & \text{o.w.} \end{cases}$

$Z \overset{\text{def}}{=} n - \sum_{R \in [n]^{k}} X_R - \sum_{B \in [n]^{\ell}} X_B$. $\mathcal{E}Z = n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{\ell}(1-p)^{\binom{\ell}{2}}$.

QED

Corollary $R(k,k) \geq \frac{k}{e}(1+o(1))\, 2^{k/2}$.

Pf:

$$R(k,k) > n - 2\binom{n}{k} 2^{-\binom{k}{2}} > n - \frac{1}{\sqrt{2\pi k}}\left(\frac{en}{k}\right)^k 2^{1-\binom{k}{2}} = n - \frac{2}{\sqrt{2\pi k}}\left(\frac{en}{k}\right)^k 2^{-\frac{k(k-1)}{2}}$$

$$> n - \left(\frac{en}{k}\right)^k 2^{-\frac{k(k-1)}{2}} \sim n - \left[2^{\frac{k}{2}}(1+o(1))\right]^k 2^{\frac{-k^2+k}{2}} \quad (\text{set } n \sim \tfrac{k}{e}(1+o(1))2^{k/2})$$

$$\sim \frac{k}{e}(1+o(1))2^{k/2}$$

QED

Bounding R(3,k) with LLL

Observation $R(3,k) > c\left(\frac{k}{\ln k}\right)^2$

pf: sample space $= \mathcal{G}_{n,p}$.

For $T \in \binom{[n]}{3}$, $S \in \binom{[n]}{k}$, let $A_T = \{G_{n,p}[T] \cong K_3\}$, $B_S = \{G_{n,p}[S] \cong \overline{K_k}\}$.

Consider dependency digraph of the events A_T and B_S:

- $A_T \to A_{T'}$, $\# \le \binom{3}{2}\binom{n}{1} \le 3n$
- $A_T \to B_{S'}$, $\# \le \binom{n}{k}$
- $B_S \to A_{T'}$. $\# \le \binom{k}{2}\binom{n}{1} \le \frac{k^2 n}{2}$
- $B_S \to B_{S'}$, $\# \le \binom{n}{k}$

Recall:

LLL says that, for events $A_1, \dots, A_n$ with dependency graph D, if $\exists\ x_1, x_2, \dots, x_n$ such that

① $0 \le x_i < 1$ $\quad i = 1, 2, 3, \dots, n$

② $P(A_i) \le x_i \prod_{\vec{ij} \text{ is an arc in } D} (1 - x_j)$, $\quad 1 \le i \le n$

then

$$P\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$$

If there exists values $x, y \in [0,1)$ and $p \in (0,1)$ such that

$P(A_T)$

$$\begin{cases} p^3 \leq x(1-x)^{3n}(1-y)^{\binom{n}{k}} \\ (1-p)^{\binom{k}{2}} \leq y(1-x)^{\frac{k^2 n}{2}}(1-y)^{\binom{n}{k}} \end{cases}$$

$P(B_s)$

then $$P\Big(\bigcap_{T \in \left[\genfrac{}{}{0pt}{}{n}{3}\right]} \bar{A}_T \cap \bigcap_{S \in \left[\genfrac{}{}{0pt}{}{n}{k}\right]} \bar{B}_S \Big) > 0.$$

That is $R(3,k) > n$.

What is the largest $k = k(n)$ so that there exist p, y, x satisfying these condition?

Well, elementary (but tedious) analysis (and a free weekend!) give that the best choice is achieved at

$$p = c_1 n^{-\frac{1}{2}}, \quad k = c_1 n^{\frac{1}{2}} \ln n, \quad x = c_3 n^{-\frac{3}{2}} \text{ and } y = \frac{c_4}{\exp\{n^{\frac{1}{2}} \ln^2 n\}}$$

We have $k = c_1 n^{\frac{1}{2}} \ln n \Rightarrow \left\langle \begin{matrix} k^2 = c_1^2 n \ln^2 n \\ \ln k \geq \frac{1}{2} \ln n \end{matrix} \right\rangle \Longrightarrow n \geq c \frac{k^2}{\ln^2 k}$

Therefore $R(3, k) > c\left(\frac{k}{\ln k}\right)^2$.

QED

NO.

DATE

spencer

A Story of Joel H. Spencer:

The values $R(3,3)$ and $R(4,4)$ were found by Greenwood and Gleason in 1955. As Gleason was my advisor, I once spent a weekend puzzling over $R(5,5)$ and then asked him for advice. He was quite clear: "Don't work on it!" Behind their elegant paper lay untold hours of calculation in attempts at improvement. The Law of Small Numbers was at work—simple patterns for k small disappear when k gets too large for easy calculation. Indeed, in the two decades since that advice, even $R(4,5)$ has remained a mystery. (There has been more success with $R(3,k)$ with $3 \leq k \leq 9$ now known.)

Erdös asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5,5)$ or they will destroy our planet. In that case, he claims, we should marshall all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6,6)$. In that case, he believes, we should attempt to destroy the aliens.

Summary

For lower bound of $R(3,k)$:

- Basic probabilistic method gave "Nothing".
- The alteration method gave $R(3,k) > k^{\frac{3}{2}+o(1)}$.
- The LLL gave $R(3,k) > c\left(\frac{k}{\ln k}\right)^2 = k^{2+o(1)}$.
- Kim (1995) gave $R(3,k) > c\frac{k^2}{\ln k}$.

For lower bound of $R(4,k)$:

- Basic probabilistic method gave $R(4,k) > c\left(\frac{k}{\ln k}\right)^{\frac{3}{2}} = k^{\frac{3}{2}+o(1)}$
- The alteration method gave $R(4,k) > c\left(\frac{k}{\ln k}\right)^2 = k^{2+o(1)}$
- The LLL gave $R(4,k) > k^{\frac{5}{2}+o(1)}$.

Lovász Local Lemma : special case

Thm (Lovász's exercise book) Let $\vec{G}=(V,\vec{E})$ be a dependency digraph of the events $A_1,\dots,A_n$ with $d^+(i)\le d\ \forall i$.

If $P_r(A_i)\le\frac{1}{4d}\ \forall i$, then $P_r(\bigcap_{i=1}^n \bar{A}_i)>0$.

pf: First, we prove $\overbrace{P(A_1|\bar{A}_2\bar{A}_3\cdots\bar{A}_n)}^{\text{LHS}}\le\frac{1}{2d}$ by induction on n. $N^+(1)\overset{\text{assumption}}{=}\{2,\cdots,\ell\}$

$$\text{LHS}=\frac{P(A_1\bar{A}_2\cdots\bar{A}_\ell\mid\bar{A}_{\ell+1}\cdots\bar{A}_n)}{P(\bar{A}_2\bar{A}_3\cdots\bar{A}_\ell\mid\bar{A}_{\ell+1}\dots\bar{A}_n)}\le\frac{P(A_1|\bar{A}_{\ell+1}\cdots\bar{A}_n)}{\underbrace{1-P(A_2+A_3+\cdots+A_\ell|\bar{A}_{\ell+1}\cdots\bar{A}_n)}_{@}}=\frac{P(A_1)}{@}\le\frac{\frac{1}{4d}}{@}$$

$$\le\frac{1/4d}{1-\sum_{i=2}^{\ell}P_r(A_i|\bar{A}_{\ell+1}\bar{A}_{\ell+2}\cdots\bar{A}_n)}\le\frac{1/4d}{1-(\ell-1)\frac{1}{2d}}\overset{@}{=}\frac{1/4d}{1-d^+(1)\frac{1}{2d}}\le\frac{1/4d}{1-d\frac{1}{2d}}=\frac{1}{2d}.$$

It turns out that $P_r(\bigcap_{i=1}^n\bar{A}_i)=P(\bar{A}_1|\bar{A}_2\cdots\bar{A}_n)\,P(\bar{A}_2\cdots\bar{A}_n)\ge(1-\frac{1}{2d})\,P(\bar{A}_2\bar{A}_3\cdots\bar{A}_n)>0$

(this is proved by induction on n, under the induction hypothesis $P(\bar{A}_2\bar{A}_3\cdots\bar{A}_n)>0$)

QED

K-satisfiability problem

K-CNF = a Boolean formula s.t. each clause has exactly k distinct literals.

$(x_1 \vee \neg x_1 \vee x_2 \vee x_3)(x_3 \vee \neg x_2 \vee x_1 \vee x_4)$

$(x_1 + \overline{x_1} + x_2 + x_3)(x_3 + \overline{x_2} + x_1 + x_4)$

KSAT: Given a Boolean formula in K-CNF is there a satisfying truth assignment to the input variables?

$(x+y)(\overline{x}+z)$ is satisfiable

$(x+z)(\overline{x})(\overline{z})$ is not satisfiable

Thm: If no variable in a KSAT formula appears in more than $\frac{2^k}{4k}$ clauses, then the formula has a satisfying assignment.

pf: Consider a random assignment to the variables.

Assume the K-CNF has the form $x_{out} = \bigvee_{i=1}^{m} \underbrace{\left(\bigwedge_{j=1}^{k} l_{ij}\right)}_{\text{the } i\text{th clause}}$.

Let $E_i \overset{def}{=} \{\, i\text{th clause is NOT satisfied} \,\}$ $i=1,2,\dots,m.$

Note that E_i → clauses containing x_1 ⋮ → clauses containing x_k (E_i: $(x_1+x_2+\dots+x_k)$) $\Big\} \le k \cdot \frac{2^k}{4k} = \underbrace{2^{k-2}}_{d}$

And $\Pr(E_i)\,4d \le 2^{-k} \cdot 4 \cdot 2^{k-2} = 1$. The Thm follows from LLL.

QED

2-coloring hypergraph with LLL

Thm Let $\mathcal{H}=(\mathcal{V},\mathcal{E})$ be a hypergraph s.t.

(1) $\forall e\in\mathcal{E} \Rightarrow |e|\geq n$

(2) $\forall e\in\mathcal{E} \Rightarrow \#\{f\in\mathcal{E} : f\cap e\neq\phi, f\neq e\}\leq d$.

If $2e(d+1)\leq 2^n$ then $\mathcal{H}$ is 2-colorable.

pf: Let $\mathcal{V}=\{1,2,3,4,5,\cdots,t\}$

Let $X_1, X_2, \cdots X_t \overset{iid}{\sim} B(1,\frac{1}{2})$

Let $A_e=\{(\forall v\in e: X_v=1) \text{ or } (\forall v\in e: X_v=0)\}$ for $\forall e\in\mathcal{E}$.

Note that

$$P(A_e)\leq 2P(\forall v\in e: X_v=1)\leq 2\left(\frac{1}{2}\right)^n\leq \frac{1}{e(d+1)}.$$

Therefore $P\left(\bigcap_{e\in\mathcal{E}}\overline{A_e}\right)>0$.

QED