Conjecture

Conjecture 5.5.2 p70 For every d-regular digraph $\vec{G}$, we have

$$\mathrm{dla}(\vec{G}) = d+1.$$

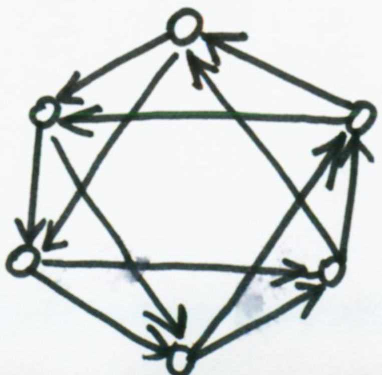
Note: Nakayama & Peroche (1987)

Dilinear arboricity $dla(\vec{G})$

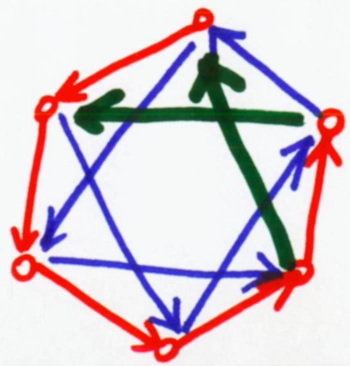
- d-regular digraph = a digraph with $d^+(x) = d^-(x)$ $\forall x \in V$.

- linear directed forest = a digraph in which every component is a dipath

- $dla(\vec{G})$ = the minimum number of linear directed forest in $\vec{G}$ whose union is the set of all arcs of $\vec{G}$.



2-regular digraph



3 linear directed forest (blue, red, green)

An Observation

Lemma A let $G=(V,E)$ be a d-regular digraph with $E = E_1 + E_2 + \cdots + E_r$ having $|E_i| \geq 2e(4d-2)$. $i=1,2,\cdots,r$.

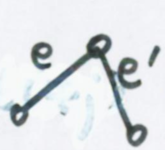
Then $\exists$ a matching M in G s.t. $|M \cap E_i| = 1$ for $i=1,2,\ldots,r$.

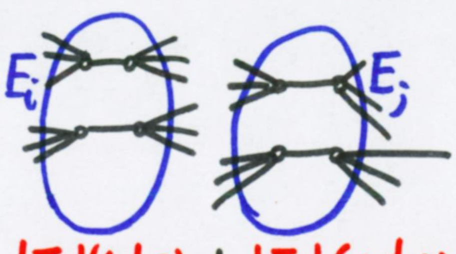
pf: $X_1, \cdots, X_r$ are independent.

$\forall i: P_r(X_i = e) = 1/|E_i|$ for any edge $e \in E_i$.

let $M \stackrel{def}{=} \{X_1, X_2, \cdots, X_r\}$. For two distinct edges

define events $A_{\{e,e'\}} = \{e \in M$ and $e' \in M\}$. Note that $e, e' \in E_i \Rightarrow P_r(A_{\{e,e'\}})=$

If $e \in E_i$, $e' \in E_j$ &  then $P(A_{\{e,e'\}}) = \dfrac{1}{|E_i||E_j|}$.
$i \neq j$



The dependency digraph of $A_{\{e,e'\}}$'s has max. degree $<$ $|E_i|(4d-2) + |E_j|(4d-2)$.

$$P(A_{\{e,e'\}})\, e\,(d+1) \leq \frac{1}{|E_i||E_j|}\, e\,(4d-2)\,(|E_i|+|E_j|) \leq e\,(4d-2)\frac{2}{2e(4d-2)} = 1.$$

QED

dla(G) conjecture holds for digraphs with large digirth

Thm 5.5.4. If $G=(V,E)$ is a d-regular digraph with digirth $g \geq 8ed$. Then $dla(G)=d+1$.

pf: "$\geq$" $(|V|-1)\, dla(G) \geq |E| = d|V| \Rightarrow dla(G) > d$

"$\leq$" Note that G has d edge-disjoint 1-regular spanning subdigraphs $F_1, F_2, \cdots, F_d$, say $\{F_1, F_2, \cdots, F_d\} = \{C_1, C_2, \cdots, C_r\}$.

edge disjoint dicycles

Note that $E = E_{C_1} + E_{C_2} + \cdots + E_{C_r}$ having $|E_{C_i}| \geq 8ed \geq 2e(4d-2)$, $\forall i$.

lemmA $\Longrightarrow \exists$ a matching M of G s.t. $|M \cap E_i| = 1$ $\forall i$.

Thus $M, F_1 \setminus M, F_2 \setminus M, \cdots, F_d \setminus M$ are $d+1$ l. directed forest of G and hence $dla(G) \leq d+1$.

QED

Chernoff's inequality (I)

Thm A $X_1, X_2, \ldots, X_n$ are independent rvs s.t. $X_i \sim B(1, p_i)\ \forall i$. Then

$$\varepsilon > 0 \Rightarrow P(S_n > (1+\varepsilon)\, \mathcal{E} S_n) \leq \left(e^{\varepsilon}/(1+\varepsilon)^{1+\varepsilon}\right)^{\mathcal{E} S_n}$$ where $S_n = \sum_{i=1}^{n} X_i$.

$$0 < \varepsilon < 1 \Rightarrow P(S_n < (1-\varepsilon)\, \mathcal{E} S_n) \leq \left(e^{-\varepsilon}/(1-\varepsilon)^{1-\varepsilon}\right)^{\mathcal{E} S_n}.$$

Pf: "$\varepsilon > 0$" $\mu \overset{\text{def}}{=} \mathcal{E} S_n$, $c \overset{\text{def}}{=} \ln(1+\varepsilon)$. Note that $\mu = \sum_{i=1}^{n} p_i$.

$$P(S_n > (1+\varepsilon)\mu) = P(e^{cS_n} > e^{c(1+\varepsilon)\mu}) \leq e^{-c(1+\varepsilon)\mu}\, \mathcal{E} e^{cS_n}$$

$$= e^{-c(1+\varepsilon)\mu} \prod_{i=1}^{n} \mathcal{E} e^{cX_i} = \underbrace{(1+\varepsilon)^{-(1+\varepsilon)\mu}}_{*} \prod_{i=1}^{n} \mathcal{E}(1+\varepsilon)^{X_i}$$

$$= * \prod_{i=1}^{n} \{(1+\varepsilon)p_i + (1-p_i)\} \leq * \prod_{i=1}^{n} e^{\varepsilon p_i} = * e^{\varepsilon\mu} = RHS.$$

For $0 < \varepsilon < 1$, let $c = -\ln(1-\varepsilon)$ and using the same argument as above.
Consider $P_r(e^{-S_n} > e^{-c(1-\varepsilon)\mu})$ to get the bound on the lower tail.

QED

Chernoff's inequality (II)

Thm B $X_1, X_2, \cdots X_n$ were defined in Thm A and $0 < \varepsilon < 1$.

Then $\Pr\{ S_n < (1-\varepsilon) ES_n \} \le e^{-\frac{\varepsilon^2 ES_n}{2}}$ and

$$\Pr\{ S_n > (1+\varepsilon) ES_n \} \le e^{-\frac{\varepsilon^2 ES_n}{3}}.$$

Pf: Note that $(1-\varepsilon)^{1-\varepsilon} > e^{-\varepsilon + \frac{\varepsilon^2}{2}}$. why?

Thm A $\Rightarrow$ LHS $\le \left(\frac{e^{-\varepsilon}}{(1-\varepsilon)^{1-\varepsilon}}\right)^{\mu} \le e^{-\frac{\varepsilon^2 \mu}{2}}$, where $\mu = ES_n$.

Note that $(1+\varepsilon)\ln(1+\varepsilon) \ge \varepsilon + \frac{\varepsilon^2}{3}$, thus $(1+\varepsilon)^{(1+\varepsilon)} \ge e^{\varepsilon + \frac{\varepsilon^2}{3}}$.

Thm A $\Rightarrow$ LHS $\le \left(\frac{e^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}}\right)^{\mu} \le e^{-\frac{\varepsilon^2 \mu}{3}}$.

QED

Note Boole's inequality $\Rightarrow \Pr\{ |S_n - ES_n| > \varepsilon ES_n \} \le 2e^{-\frac{\varepsilon^2 \mu}{3}}$.