

Chernoff's inequality (III)

Thm C $X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} P_r(X_i = \pm 1) = \frac{1}{2}$. $S_n \stackrel{\text{def}}{=} \sum_{i=1}^n X_i$.

Then for any $t \geq 0$, $P_r(S_n \geq t) < e^{-\frac{t^2}{2n}}$.

Pf: $P_r(S_n \geq t) = P_r(e^{xS_n} \geq e^{xt})$ ($x > 0$ not yet determined)

$$\leq \prod_{i=1}^n E e^{xX_i} / e^{xt} = \left(\frac{e^x + e^{-x}}{2} \right)^n / e^{xt}$$

$$= \left(\frac{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots + 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\dots}{2} \right)^n / e^{xt} = \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots \right)^n / e^{xt}$$

$$< e^{\frac{x^2}{2}n} / e^{xt} = e^{\frac{nx^2}{2} - tx}. \quad \text{Set } x = \frac{t}{n} \text{ to get RHS.}$$

Note: $P_r(|S_n| \geq t) < 2e^{-\frac{t^2}{2n}}$.

Exercise (E)

problem 10 show that if $X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} B(1, p)$,
 $\lambda = np$ and $S_n = \sum_{i=1}^n X_i$. Then

$$P_r(S_n \geq t + \varepsilon S_n) \leq \left(\frac{\lambda}{\lambda + t}\right)^{\lambda + t} \left(\frac{n - \lambda}{n - \lambda - t}\right)^{n - \lambda - t},$$

for any $0 \leq t \leq n - \lambda$.

problem 11 show that $(1 - \varepsilon)^{1 + \varepsilon} \geq e^{-\varepsilon + \frac{\varepsilon^2}{2}}$
and $(1 + \varepsilon)^{1 + \varepsilon} \geq e^{\varepsilon + \frac{\varepsilon^2}{3}}$, for any $0 < \varepsilon < 1$.

Lemma 5.5.5^{P71} G is a d -regular digraph with d sufficiently large and $p \in \mathbb{Z}^+$. There exists $\varphi: V_G \rightarrow \{0, 1, 2, \dots, p-1\}$ s.t.

$$|N_c^\pm(x) - \frac{d}{p}| \leq 3\sqrt{\frac{d}{p}}\sqrt{\ln d} \text{ for all } x \in V_G \text{ and } c \in \{0, 1, 2, \dots, p-1\},$$

where $N_c^\pm(x) = |\{y \in N^\pm(x) : \varphi(y) = c\}|$.

pf: $\mathbb{P}\{X_v = c\} = \frac{1}{p}$ for $c = 0, 1, 2, \dots, p-1$, $\forall v \in V_G$.

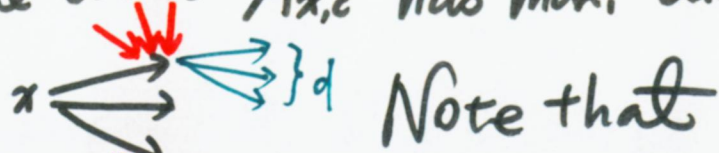
$$A_{x,c}^\pm \stackrel{\text{def}}{=} \{|N_c^\pm(x) - \frac{d}{p}| > \underbrace{3\sqrt{\frac{d}{p}}\sqrt{\ln d}}_{\varepsilon'}\}. \text{ Note that } \frac{d}{p} = \varepsilon \star$$

$$\mathbb{P}(A_{x,c}^+) = \mathbb{P}\left(|\sum_{v \in N^+(x)} I_{\{X_v=c\}} - \frac{d}{p}| > \varepsilon'\right) = \mathbb{P}\left(|\star - \frac{d}{p}| > \varepsilon \frac{d}{p}\right), \quad \varepsilon = 3\sqrt{\frac{p}{d}}\sqrt{\ln d}$$

$$\leq 2e^{-\frac{\varepsilon^2 \varepsilon \star}{3}} = 2\exp\left\{-\frac{9 \frac{p}{d} \ln d \frac{d}{p}}{3}\right\} = \frac{2}{d^3}. \text{ Hence } \mathbb{P}(A_{x,c}^\pm) \leq \frac{2}{d^3}.$$

chernoff's ineq. (II) The dependency digraph of the events $A_{x,c}^\pm$ has max. outdegree

$$\leq \{d(d-1) + d^2\}p \leq 2d^2p.$$



Note that

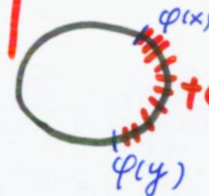
$$\mathbb{P}(A_{x,c}^\pm) e^{\{2d^2p+1\}} \leq \frac{2}{d^3} e^{\{2d^2p+1\}} < 1, \quad \text{since } d \text{ is sufficiently large. QED}$$

Thm $\exists$ constant $c > 0$ s.t. for $\forall$ d -regular digraph G , ↖ large enough

$$dla(G) \leq d + c d^{3/4} (\ln d)^{1/2}$$

pf: $\exists$ a prime $p \in [10\sqrt{d}, 20\sqrt{d}]$. let φ be defined in Lemma 5.5.5. and

$$E_c \stackrel{\text{def}}{=} \{ \overrightarrow{xy} : \varphi(y) \equiv \varphi(x) + c \pmod{p} \} \text{ i.e.}$$



Note $E_G = E_0 + E_1 + \dots + E_{p-1}$.

let $G_c = (V_G, E_c)$ be a spanning subdigraph of G , for $c = 0, 1, 2, \dots, p-1$.

Step 1: Consider the upper bounds for $dla(G_1), dla(G_2), \dots, dla(G_{p-1})$.

Max. ⁱⁿ degree of G_c $\Delta_c^+ \leq \frac{d}{p} + 3\sqrt{\frac{d}{p}} \sqrt{\ln d}$ ($\because$ Lemma 5.5.5) ($0 < c \leq p-1$)

Suppose $t = \text{digirth}(G_c)$ i.e. $\varphi(x_1) \xrightarrow{+c} \varphi(x_2) \xrightarrow{+c} \dots \xrightarrow{+c} \varphi(x_t)$ and hence $p \mid ct$

and so $p \mid t$. Therefore $\text{digirth}(G) \geq p$.

(here we use the property that p is a prime)

pf (continued) Therefore there exists a Δ_c -regular digraph H s.t.

① G_c is a subdigraph of H ,

② $\Delta_c = \max \{ \Delta_c^+, \Delta_c^- \} \leq \frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d},$

③ $\text{digirth}(H) = \text{digirth}(G_c) \geq p \geq 8e \left\{ 4\frac{d}{p} \right\} \geq 8e\Delta_c.$

$(\because p \geq 10\sqrt{d} \Rightarrow p \geq \frac{100d}{p} \geq 8 \times 3 \left\{ 4\frac{d}{p} \right\}) \quad (\because d \gg 1)$

Thm 5.5.4 $\Rightarrow d_{la}(G_c) \leq d_{la}(H) \leq \Delta_c + 1 \leq \frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d} + 1.$

So

$d_{la}(G) \leq d_{la}(G_0) + \underbrace{d_{la}(G_1) + \dots + d_{la}(G_{p-1})}_{\wedge}$

$(p-1) \left\{ \frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d} + 1 \right\}$

pf (continued)

Step 2: Consider the upper bound for $dla(G_0)$. We have

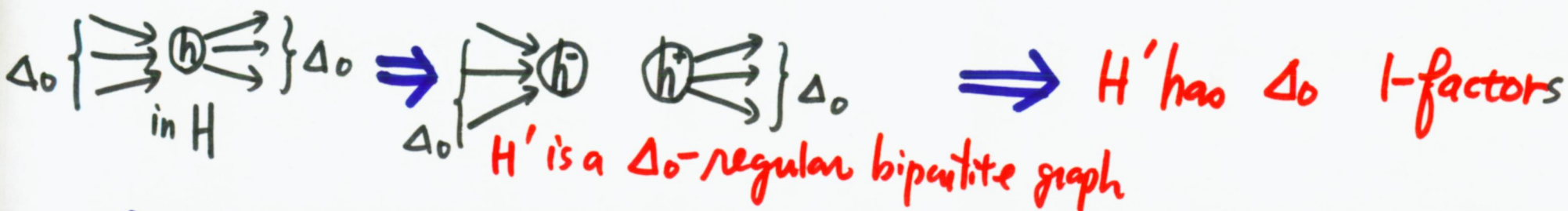
$$\max_{\text{in}, \text{out}} \text{degree of } G_0 \quad \Delta_0^\pm \leq \frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d}.$$

Therefore $\exists$ a Δ_0 -regular digraph H s.t.

① G_0 is a subdigraph of H .

$$\textcircled{2} \quad \Delta_0 = \max\{\Delta_0^+, \Delta_0^-\} \leq \frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d}$$

(note: here we can not control $\text{digirth}(G_0)$)



$$\Rightarrow H \text{ has } \Delta_0 \text{ 2-factors} \Rightarrow dla(G_0) \leq dla(H) \leq 2\Delta_0 \leq 2\frac{d}{p} + 6\sqrt{\frac{d}{p}}\sqrt{\ln d}$$

(decompose each 2-factor into 2 linear digforests)

pf (continued)

$$dla(G) \leq dla(G_0) + dla(G_1) + dla(G_2) + \dots + dla(G_{p-1})$$

$$\leq 2\frac{d}{p} + 6\sqrt{\frac{d}{p}}\sqrt{\ln d} + (p-1)\left\{\frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d} + 1\right\}$$

why?

$$\leq d + cd^{\frac{3}{4}}(\ln d)^{\frac{1}{2}} \text{ for some constant by using the}$$

fact that $p \in [10\sqrt{d}, 20\sqrt{d}]$ and d sufficiently large.

QED