Chernoff's inequality (I)

Thm A $X_1, X_2, \ldots, X_n$ are independent rvs s.t. $X_i \sim B(1, p_i)\ \forall i$. Then

$$\varepsilon > 0 \Rightarrow P(S_n > (1+\varepsilon)\,\mathcal{E}S_n) \leq \left(e^{\varepsilon}/(1+\varepsilon)^{1+\varepsilon}\right)^{\mathcal{E}S_n}, \text{ where } S_n = \sum_{i=1}^n X_i.$$

$$0 < \varepsilon < 1 \Rightarrow P(S_n < (1-\varepsilon)\,\mathcal{E}S_n) \leq \left(e^{-\varepsilon}/(1-\varepsilon)^{1-\varepsilon}\right)^{\mathcal{E}S_n}.$$

Pf: "$\varepsilon > 0$" $\mu \overset{def}{=} \mathcal{E}S_n$, $c \overset{def}{=} \ln(1+\varepsilon)$. Note that $\mu = \sum_{i=1}^n p_i$.

$$P(S_n > (1+\varepsilon)\mu) = P(e^{cS_n} > e^{c(1+\varepsilon)\mu}) \leq e^{-c(1+\varepsilon)\mu}\,\mathcal{E}e^{cS_n}$$

$$= e^{-c(1+\varepsilon)\mu} \prod_{i=1}^n \mathcal{E}e^{cX_i} = \underbrace{(1+\varepsilon)^{-(1+\varepsilon)\mu}}_{\star} \prod_{i=1}^n \mathcal{E}(1+\varepsilon)^{X_i}$$

$$= \star \prod_{i=1}^n \{(1+\varepsilon)p_i + (1-p_i)\} \leq \star \prod_{i=1}^n e^{\varepsilon p_i} = \star\, e^{\varepsilon\mu} = RHS.$$

For $0<\varepsilon<1$, let $c = -\ln(1-\varepsilon)$ and using the same argument as above. Consider $P_r(e^{-S_n} > e^{-c(1-\varepsilon)\mu})$ to get the bound on the lower tail.

QED

Chernoff's inequality (II)

Thm B $X_1, X_2, \cdots X_n$ were defined in Thm A and $0<\varepsilon<1$.

Then $\Pr\{ S_n < (1-\varepsilon)\mathcal{E} S_n \} \leq e^{-\frac{\varepsilon^2 \mathcal{E} S_n}{2}}$ and

$$\Pr\{ S_n > (1+\varepsilon)\mathcal{E} S_n \} \leq e^{-\frac{\varepsilon^2 \mathcal{E} S_n}{3}}.$$

Pf: Note that $(1-\varepsilon)^{1-\varepsilon} > e^{-\varepsilon + \frac{\varepsilon^2}{2}}$. why?

Thm A $\Rightarrow$ LHS $\leq \left(\frac{e^{-\varepsilon}}{(1-\varepsilon)^{1-\varepsilon}}\right)^{\mu} \leq e^{-\frac{\varepsilon^2 \mu}{2}}$, where $\mu = \mathcal{E} S_n$.

Note that $(1+\varepsilon)\ln(1+\varepsilon) \geq \varepsilon + \frac{\varepsilon^2}{3}$, thus $(1+\varepsilon)^{(1+\varepsilon)} \geq e^{\varepsilon + \frac{\varepsilon^2}{3}}$.

Thm A $\Rightarrow$ LHS $\leq \left(\frac{e^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}}\right)^{\mu} \leq e^{-\frac{\varepsilon^2 \mu}{3}}$.

QED

Note Boole's inequality $\Rightarrow \Pr\{ |S_n - \mathcal{E} S_n| > \varepsilon \mathcal{E} S_n \} \leq 2e^{-\frac{\varepsilon^2 \mu}{3}}$.

Chernoff's inequality (III)

Thm C $X_1, X_2, \cdots X_n \overset{iid}{\sim}$ $\Pr(X_1 = \pm 1) = \frac{1}{2}$. $S_n \overset{def}{=\!=} \sum_{i=1}^{n} X_i$.

Then for any $t \geq 0$, $\Pr(S_n \gtreqless t) < e^{-\frac{t^2}{2n}}$.

Pf: $\Pr(S_n \geq t) = \Pr(e^{xS_n} \geq e^{xt})$ ($x > 0$ not yet determined)

$$\leq \prod_{i=1}^{n} \mathcal{E} e^{xX_i} / e^{xt} = \left(\frac{e^x + e^{-x}}{2}\right)^n / e^{xt}$$

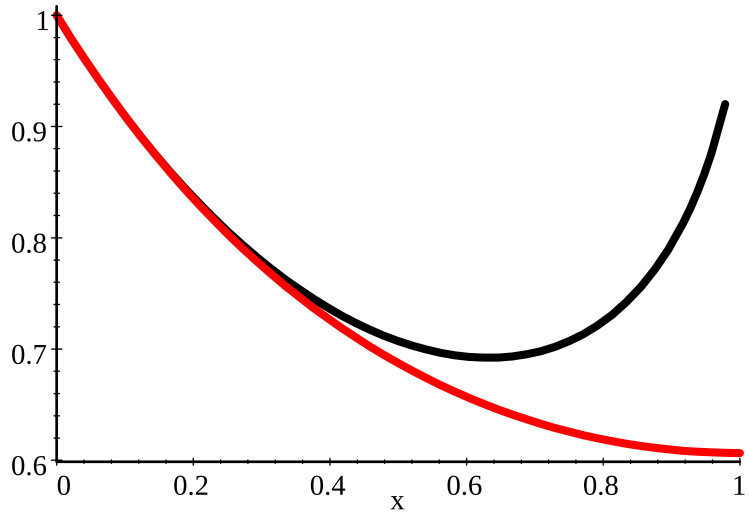
$$= \left(\frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad + 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots}{2}\right)^n / e^{xt} = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)^n / e^{xt}$$

$$< e^{\frac{x^2}{2}n} / e^{xt} = e^{\frac{nx^2}{2} - tx}.$$

Set $x = \frac{t}{n}$ to get RHS.

Note: $\Pr(|S_n| \geq t) < 2e^{-\frac{t^2}{2n}}$.

$$(1-\varepsilon)^{1-\varepsilon} \ge e^{-\varepsilon+\frac{\varepsilon^2}{2}}$$



$$\Uparrow$$

$$0<\varepsilon<1$$

$$\Downarrow$$

$$(1+\varepsilon)^{1+\varepsilon} \ge e^{\varepsilon+\frac{\varepsilon^2}{3}}$$