Conjecture

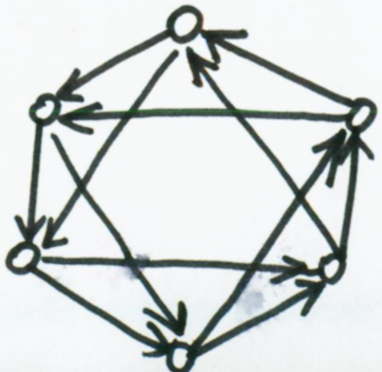
Conjecture 5.5.2 p70 For every d-regular digraph $\vec{G}$, we have

$$\mathrm{dla}(\vec{G}) = d+1.$$

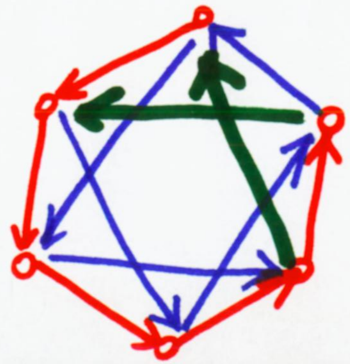
Note: Nakayama & Peroche (1987)

Dilinear arboricity $dla(\vec{G})$

- d-regular digraph = a digraph with $d^+(x) = d^-(x)$ $\forall x \in V$.
- linear directed forest = a digraph in which every component is a dipath
- $dla(\vec{G})$ = the minimum number of linear directed forest in $\vec{G}$ whose union is the set of all arcs of $\vec{G}$.



2-regular digraph



3 linear directed forest (blue, red, green)

An Observation

Lemma A let $G=(V,E)$ be a d-regular digraph with

$E = E_1 + E_2 + \cdots + E_r$ having $|E_i| \geq 2e(4d-2)$, $i=1,2,\cdots,r$.

Then $\exists$ a matching M in G s.t. $|M \cap E_i| = 1$ for $i=1,2,\cdots,r$.

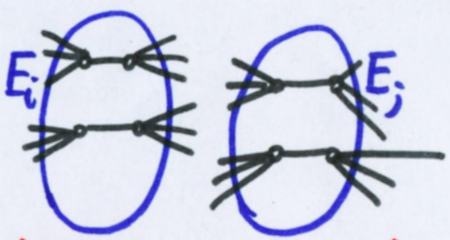
pf: $X_1, \cdots, X_r$ are independent.

$\forall i$: $\Pr(X_i = e) = 1/|E_i|$ for any edge $e \in E_i$.

let $M \stackrel{\text{def}}{=} \{X_1, X_2, \cdots, X_r\}$. For two distinct edges e, e' with

define events $A_{\{e,e'\}} = \{e \in M \text{ and } e' \in M\}$. Note that $e, e' \in E_i \Rightarrow \Pr(A_{\{e,e'\}})=0$

If $e \in E_i$, $e' \in E_j$ ($i \neq j$) then $P(A_{\{e,e'\}}) = \frac{1}{|E_i||E_j|}$.



The dependency digraph of $A_{\{e,e'\}}$'s has max. degree $\leq |E_i|(4d-2) + |E_j|(4d-2)$.

$$P(A_{\{e,e'\}})\, e\,(d+1) \leq \frac{1}{|E_i||E_j|} e\,(4d-2)\,(|E_i| + |E_j|) \leq e\,(4d-2)\frac{2}{2e(4d-2)} = 1.$$

QED

dla(G) conjecture holds for digraphs with large digirth

Thm 5.5.4. If $G=(V,E)$ is a d-regular digraph with digirth $g \geq 8ed$. Then $dla(G)=d+1$.

pf: "$\geq$" $(|V|-1)\, dla(G) \geq |E| = d|V| \Rightarrow dla(G) > d$

"$\leq$" Note that G has d edge-disjoint 1-regular spanning subdigraphs $F_1, F_2, \cdots, F_d$, say $\{F_1, F_2, \cdots, F_d\} = \{C_1, C_2, \cdots, C_r\}$.

edge disjoint dicycles

Note that $E = E_{C_1} + E_{C_2} + \cdots + E_{C_r}$ having $|E_{C_i}| \geq 8ed \geq 2e(4d-2)$, $\forall i$.

LemmA $\Rightarrow \exists$ a matching M of G s.t. $|M \cap E_i| = 1$ $\forall i$.

Thus $M, F_1 \setminus M, F_2 \setminus M, \cdots, F_d \setminus M$ are $d+1$ l. directed forest of G and hence $dla(G) \leq d+1$.

QED

a

Lemma 5.5.5 [P71] G is a d-regular digraph with d sufficiently large and $p \in \mathbb{Z}^+$. There exists $\varphi: V_G \longrightarrow \{0, 1, 2, \cdots, p-1\}$ s.t.

$$\left| N_c^{\pm}(x) - \frac{d}{p} \right| \leq 3\sqrt{\frac{d}{p}}\sqrt{\ln d} \text{ for all } x \in V_G \text{ and } c \in \{0,1,2,\cdots,p-1\},$$

where $N_c^{\pm}(x) = \#\{y \in N^{\pm}(x) : \varphi(y) = c\}$.

Pf: $P_r\{X_v = c\} = \frac{1}{p}$ for $c = 0, 1, 2, \cdots, p-1$, $\forall v \in V_G$.

$\varepsilon \in (0,1)$

$$A_{x,c}^{\pm} \overset{def}{=} \left\{ \left| N_c^{\pm}(x) - \frac{d}{p} \right| > \underbrace{3\sqrt{\frac{d}{p}}\sqrt{\ln d}}_{\varepsilon'} \right\}.$$ Note that $\frac{d}{p} = \varepsilon^*$

$$P_r(A_{x,c}^{+}) = P_r\left(\left| \underbrace{\sum_{v \in N^+(x)} I_{\{X_v = c\}}}_{*} - \frac{d}{p} \right| > \varepsilon' \right) = P_r\left(\left| * - \frac{d}{p} \right| > \varepsilon \frac{d}{p} \right),\quad \varepsilon = 3\sqrt{\frac{p}{d}}\sqrt{\ln d}$$

$$\leq 2e^{-\frac{\varepsilon^2 \varepsilon^*}{3}} = 2\exp^*\left\{ -\frac{9\frac{p}{d}\ln d \, \frac{d}{p}}{3} \right\} = \frac{2}{d^3}.$$ Hence $P(A_{x,c}^{-}) \leq \frac{2}{d^3}$.

chernoff's ineq. (II)

The dependency digraph of the events $A_{x,c}^{\pm}$ has max. outdegree $\leq \{d(d-1)+d^2\}p \leq 2d^2p$.

Note that

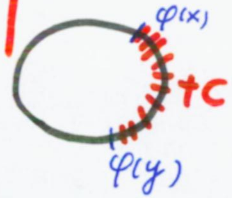
$$P_r(A_{x,c}^{\pm})e\{2d^2p+1\} \leq \frac{2}{d^3}e\{2d^2p+1\} < 1,$$ since d is sufficiently large. **QED**

Thm $\exists$ constant $c>0$ s.t. for $\forall$ d-regular digraph G, (large enough: d)

$$dla(G) \leq d + c\, d^{3/4} (\ln d)^{\frac{1}{2}}.$$

pf: $\exists$ a prime $p \in [10\sqrt{d}, 20\sqrt{d}]$. let φ be defined in Lemma 5.5.5. and

$E_c \overset{\text{def}}{=} \{ \overrightarrow{xy} : \varphi(y) \equiv \varphi(x) + c \mod p \}$ i.e.



. Note $E_G = E_0 + E_1 + \cdots + E_{p-1}$.

let $G_c = (V_G, E_c)$ be a spanning subdigraph of G, for $c = 0, 1, 2, \cdots, p-1$.

Step 1: Consider the upper bounds for $dla(G_1), dla(G_2), \cdots, dla(G_{p-1})$.

max. out/in degree of G_c Δ_c^+ / Δ_c^- $\leq \frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d}$ ($\because$ Lemma 5.5.5) $(0 < c \leq p-1)$

Suppose $t = \text{digirth}(G_c)$ i.e. $\varphi(x_1) \xrightarrow{+c} \varphi(x_2) \xrightarrow{+c} \cdots\cdots \xrightarrow{+c} \varphi(x_t) \xrightarrow{+c} \varphi(x_1)$ (mod p) and hence $p \mid ct$

and so $p \mid t$. Therefore $\text{digirth}(G) \geq p$.

(here we use the property that p is a prime)

Pf (continued) Therefore there exists a Δ_c-regular digraph H s.t.

① G_c is a subdigraph of H,

② $\Delta_c = \max\{\Delta_c^+, \Delta_c^-\} \le \frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d}$,

③ $\text{digirth}(H) = \text{digirth}(G_c) \ge p \ge 8e\{4\frac{d}{p}\} \underset{(\because d >> 1)}{\ge} 8e\Delta_c$.

$(\because p \ge 10\sqrt{d} \Rightarrow p \ge \frac{100d}{p} \ge 8 \times 3\{4\frac{d}{p}\})$

Thm 5.5.4 $\Rightarrow dla(G_c) \le dla(H) \le \Delta_c + 1 \le \frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d} + 1$.

So

$$dla(G) \le dla(G_0) + \underbrace{dla(G_1) + \cdots + dla(G_{p-1})}_{\le (p-1)\left\{\frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d} + 1\right\}}.$$

Pf (continued)

Step 2: Consider the upper bound for $dla(G_0)$. We have

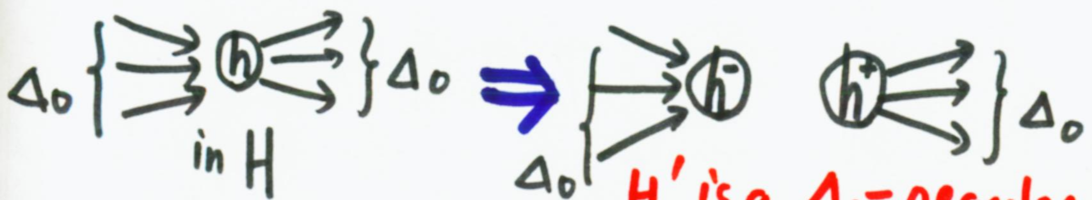
max. in/out degree of G_0 $\begin{matrix}\Delta_0^-\\ \Delta_0^+\end{matrix} \leq \frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d}$.

Therefore $\exists$ a Δ_0-regular digraph H s.t.

① G_0 is a subdigraph of H.

② $\Delta_0 = \max\{\Delta_0^+, \Delta_0^-\} \leq \frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d}$

(note: here we can not control digirth(G_0))



$\Rightarrow$ H' has Δ_0 1-factors

H' is a Δ_0-regular bipartite graph

$\Rightarrow$ H has Δ_0 2-factors $\Rightarrow$ $dla(G_0) \leq dla(H) \leq 2\Delta_0 \leq 2\frac{d}{p} + 6\sqrt{\frac{d}{p}}\sqrt{\ln d}$

(decompose each 2-factor into 2 linear diforest)

pf (continued)

$$\mathrm{dla}(G) \leq \mathrm{dla}(G_0) + \mathrm{dla}(G_1) + \mathrm{dla}(G_2) + \cdots + \mathrm{dla}(G_{p-1})$$

$$\leq 2\frac{d}{p} + 6\sqrt{\frac{d}{p}}\sqrt{\ln d} + (p-1)\left\{\frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d} + 1\right\}$$

why?

$$\leq d + c d^{\frac{3}{4}} (\ln d)^{\frac{1}{2}}$$ for some constant by usig the fact that $p \in [10\sqrt{d}, 20\sqrt{d}]$ and d sufficiently large.

QED