Conjecture

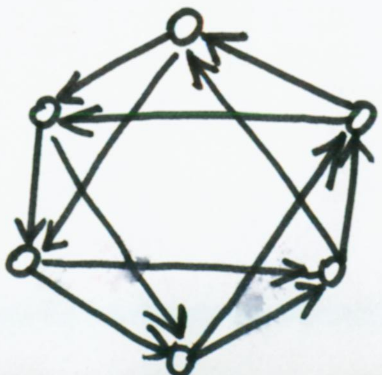
Conjecture 5.5.2 p70 For every d-regular digraph $\vec{G}$, we have

$$dla(\vec{G}) = d+1.$$

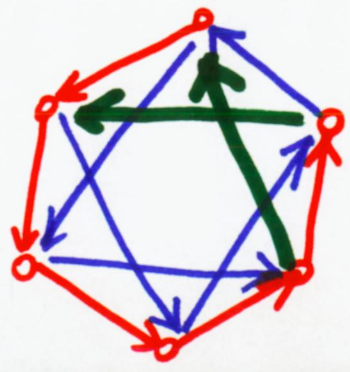
Note: Nakayama & Peroche (1987)

Dilinear arboricity $dla(\vec{G})$

- d-regular digraph = a digraph with $d^+(x)=d^-(x)$ $\forall x \in V$.
- linear directed forest = a digraph in which every component is a dipath
- $dla(\vec{G})$ = the minimum number of linear directed forest in $\vec{G}$ whose union is the set of all arcs of $\vec{G}$.



2-regular digraph



3 linear directed forest (blue, red, green)

An Observation

Lemma A Let $G=(V,E)$ be a d-regular digraph with $E = E_1 + E_2 + \cdots + E_r$ having $|E_i| \geq 2e(4d-2)$, $i=1,2,\cdots,r$.

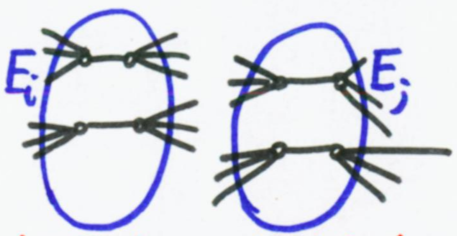
Then $\exists$ a matching M in G s.t. $|M \cap E_i| = 1$ for $i=1,2,\ldots,r$.

Pf: $X_1,\cdots,X_r$ are independent.

$\forall i: \Pr(X_i = e) = 1/|E_i|$ for any edge $e \in E_i$.

Let $M \stackrel{\text{def}}{=} \{X_1, X_2, \cdots, X_r\}$. For two distinct edges $e, e' \in E$, define events $A_{\{e,e'\}} = \{e \in M \text{ and } e' \in M\}$. Note that $e, e' \in E_i \Rightarrow \Pr(A_{\{e,e'\}}) = 0$.

If $e \in E_i$, $e' \in E_j$ $(i \neq j)$ then $P(A_{\{e,e'\}}) = \dfrac{1}{|E_i||E_j|}$.



The dependency digraph of $A_{\{e,e'\}}$'s has max. degree $<$ $|E_i|(4d-2) + |E_j|(4d-2)$.

$$P(A_{\{e,e'\}})\, e\,(d+1) \leq \frac{1}{|E_i||E_j|}\, e\,(4d-2)\,(|E_i|+|E_j|) \leq e\,(4d-2)\,\frac{2}{2e(4d-2)} = 1.$$

QED

dla(G) conjecture holds for digraphs with large digirth

Thm 5.5.4. If $G=(V,E)$ is a d-regular digraph with digirth $g \geq 8ed$.
Then $dla(G)=d+1$.

pf: "$\geq$" $(|V|-1)\,dla(G) \geq |E| = d|V| \Rightarrow dla(G) > d$

"$\leq$" Note that G has d edge-disjoint 1-regular spanning subdigraphs
$F_1, F_2, \cdots, F_d$, say $\{F_1, F_2, \cdots, F_d\} = \{C_1, C_2, \cdots, C_r\}$.
edge disjoint dicycles

Note that $E = E_{C_1} + E_{C_2} + \cdots + E_{C_r}$ having $|E_{C_i}| \geq 8ed \geq 2e(4d-2)$, $\forall i$.

LemmA $\Rightarrow \exists$ a matching M of G s.t. $|M \cap E_i| = 1$ $\forall i$.

Thus $M, F_1 \backslash M, F_2 \backslash M, \cdots, F_d \backslash M$ are $d+1$ l. directed forest of G and hence $dla(G) \leq d+1$.

QED

Chernoff's inequality (I)

Thm A $X_1, X_2, \dots, X_n$ are independent rvs s.t. $X_i \sim B(1, p_i)\ \forall i$. Then

$$\varepsilon > 0 \Rightarrow P(S_n > (1+\varepsilon)\,\mathcal{E}S_n) \le \left(e^{\varepsilon}/(1+\varepsilon)^{1+\varepsilon}\right)^{\mathcal{E}S_n}$$ where $S_n = \sum_{i=1}^n X_i$.

$$0 < \varepsilon < 1 \Rightarrow P(S_n < (1-\varepsilon)\,\mathcal{E}S_n) \le \left(e^{-\varepsilon}/(1-\varepsilon)^{1-\varepsilon}\right)^{\mathcal{E}S_n}.$$

Pf: "$\varepsilon > 0$" $\mu \overset{\text{def}}{=} \mathcal{E}S_n$, $c \overset{\text{def}}{=} \ln(1+\varepsilon)$. Note that $\mu = \sum_{i=1}^n p_i$.

$$P(S_n > (1+\varepsilon)\mu) = P(e^{cS_n} > e^{c(1+\varepsilon)\mu}) \le e^{-c(1+\varepsilon)\mu}\,\mathcal{E}e^{cS_n}$$

$$= e^{-c(1+\varepsilon)\mu} \prod_{i=1}^n \mathcal{E}e^{cX_i} = \underbrace{(1+\varepsilon)^{-(1+\varepsilon)\mu}}_{\star} \prod_{i=1}^n \mathcal{E}(1+\varepsilon)^{X_i}$$

$$= \star \prod_{i=1}^n \{(1+\varepsilon)p_i + (1-p_i)\} \le \star \prod_{i=1}^n e^{\varepsilon p_i} = \star\, e^{\varepsilon\mu} = RHS.$$

For $0 < \varepsilon < 1$, let $c = -\ln(1-\varepsilon)$ and using the same argument as above. Consider $P_r(e^{-S_n} > e^{-c(1-\varepsilon)\mu})$ to get the bound on the lower tail.

QED

Chernoff's inequality (II)

Thm B $X_1, X_2, \cdots X_n$ were defined in Thm A and $0<\varepsilon<1$.

Then $\Pr\{S_n < (1-\varepsilon)ES_n\} \le e^{-\frac{\varepsilon^2 ES_n}{2}}$ and

$$\Pr\{S_n > (1+\varepsilon)ES_n\} \le e^{-\frac{\varepsilon^2 ES_n}{3}}.$$

Pf: Note that $(1-\varepsilon)^{1-\varepsilon} > e^{-\varepsilon+\frac{\varepsilon^2}{2}}$. why?

Thm A $\Rightarrow$ LHS $\le \left(\frac{e^{-\varepsilon}}{(1-\varepsilon)^{1-\varepsilon}}\right)^{\mu} \le e^{-\frac{\varepsilon^2\mu}{2}}$, where $\mu = ES_n$.

Note that $(1+\varepsilon)\ln(1+\varepsilon) \ge \varepsilon + \frac{\varepsilon^2}{3}$, thus $(1+\varepsilon)^{(1+\varepsilon)} \ge e^{\varepsilon+\frac{\varepsilon^2}{3}}$.

Thm A $\Rightarrow$ LHS $\le \left(\frac{e^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}}\right)^{\mu} \le e^{-\frac{\varepsilon^2\mu}{3}}$.

QED

Note Boole's inequality $\Rightarrow \Pr\{|S_n - ES_n| > \varepsilon ES_n\} \le 2e^{-\frac{\varepsilon^2\mu}{3}}$.

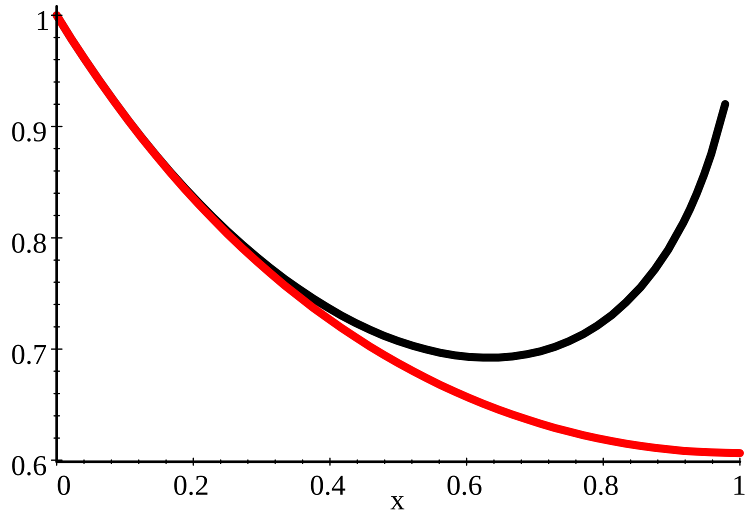
$$(1-\varepsilon)^{1-\varepsilon} \geq e^{-\varepsilon+\frac{\varepsilon^2}{2}}$$

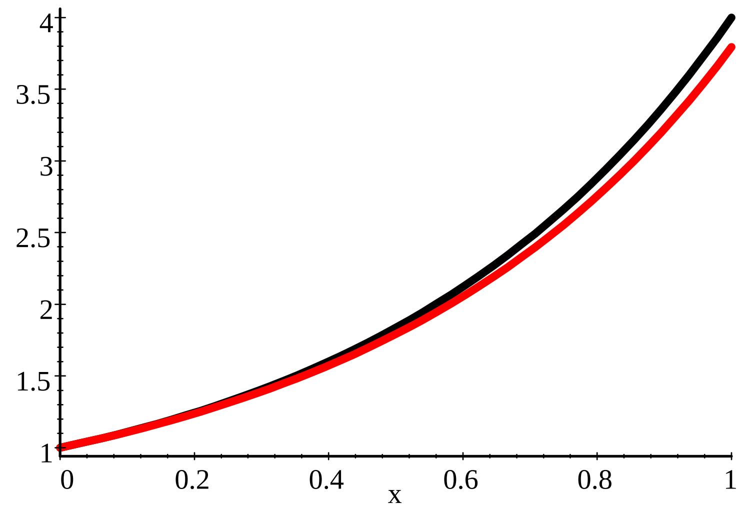
$$\Uparrow$$

$$0 < \varepsilon < 1$$

$$\Downarrow$$

$$(1+\varepsilon)^{1+\varepsilon} \geq e^{\varepsilon+\frac{\varepsilon^2}{3}}$$





Chernoff's inequality (III)

Thm C $X_1, X_2, \cdots X_n \overset{iid}{\sim} \Pr(X_1 = \pm 1) = \frac{1}{2}$. $S_n \overset{def}{=\!=} \sum_{i=1}^{n} X_i$.

Then for any $t \geq 0$, $\Pr(S_n \underset{\leqq}{\geqq} t) < e^{-\frac{t^2}{2n}}$.

Pf: $\Pr(S_n \geq t) = \Pr(e^{xS_n} \geq e^{xt})$ ($x > 0$ not yet determined)

$\leq \prod_{i=1}^{n} \mathcal{E} e^{xX_i} / e^{xt} = \left(\frac{e^x + e^{-x}}{2}\right)^n / e^{xt}$

$= \left(\frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots}{2}\right)^n / e^{xt} = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)^n / e^{xt}$

$< e^{\frac{x^2}{2}n} / e^{xt} = e^{\frac{nx^2}{2} - tx}$. Set $x = \frac{t}{n}$ to get RHS.

Note: $\Pr(|S_n| \geq t) < 2e^{-\frac{t^2}{2n}}$.

Exercise (E)

problem 10 Show that if $X_1, X_2, \cdots, X_n \overset{iid}{\sim} B(1,p)$, $\lambda = np$ and $S_n = \sum_{i=1}^{n} X_i$. Then

$$P_r\left(S_n \geq t + ES_n\right) \leq \left(\frac{\lambda}{\lambda + t}\right)^{\lambda + t} \left(\frac{n-\lambda}{n-\lambda-t}\right)^{n-\lambda-t},$$

for any $0 \leq t \leq n - \lambda$.

problem 11 Show that $(1-\varepsilon)^{1-\varepsilon} \geq e^{-\varepsilon + \frac{\varepsilon^2}{2}}$ and $(1+\varepsilon)^{1+\varepsilon} \geq e^{\varepsilon + \frac{\varepsilon^2}{3}}$, for any $0 < \varepsilon < 1$.

Lemma 5.5.5 P71 G is a d-regular digraph with d sufficiently large and $p \in \mathbb{Z}^+$. There exists $\varphi: V_G \longrightarrow \{0, 1, 2, \cdots, p-1\}$ s.t.

$$\left| N_c^{\pm}(x) - \frac{d}{p} \right| \le 3\sqrt{\frac{d}{p}}\sqrt{\ln d} \text{ for all } x \in V_G \text{ and } c \in \{0,1,2,\cdots,p-1\},$$

where $N_c^{\pm}(x) = \#\{y \in N^{\pm}(x) : \varphi(y) = c\}$.

Pf: $P_r\{X_v = c\} = \frac{1}{p}$ for $c = 0, 1, 2, \cdots, p-1$, $\forall v \in V_G$.

$A_{x,c}^{\pm} \overset{\text{def}}{=} \left\{ \left| N_c^{\pm}(x) - \frac{d}{p} \right| > \underbrace{3\sqrt{\frac{d}{p}}\sqrt{\ln d}}_{\varepsilon'} \right\}$. Note that $\varepsilon \in (0,1)$ $\quad \frac{d}{p} = \varepsilon^*$

$$P_r(A_{x,c}^{+}) = P_r\left(\left| \underbrace{\sum_{v \in N^+(x)} I_{\{X_v = c\}}}_{*} - \frac{d}{p} \right| > \varepsilon' \right) = P_r\left(\left| * - \frac{d}{p} \right| > \varepsilon \frac{d}{p} \right), \quad \varepsilon = 3\sqrt{\frac{p}{d}}\sqrt{\ln d}$$

$$\le 2e^{-\frac{\varepsilon^2 \varepsilon^*}{3}} = 2\exp\left\{ -\frac{9\frac{p}{d}\ln d \,\frac{d}{p}}{3} \right\} = \frac{2}{d^3}. \text{ Hence } P(A_{x,c}^{-}) \le \frac{2}{d^3}.$$

chernoff's ineq. (II)

The dependency digraph of the events $A_{x,c}^{\pm}$ has max. outdegree

$\le \{d(d-1) + d^2\}p \le 2d^2p$.

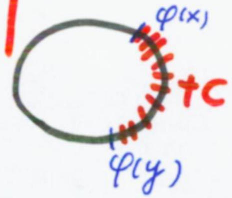
Note that

$P_r(A_{x,c}^{\pm})\, e\{2d^2p+1\} \le \frac{2}{d^3} e\{2d^2p+1\} < 1$, $d-1$ since d is sufficiently large. **QED**

Thm $\exists$ constant $c>0$ s.t. for $\forall$ d-regular digraph G, (large enough)

$$dla(G) \le d + c\,d^{3/4}(\ln d)^{\frac{1}{2}}.$$

pf: $\exists$ a prime $p \in [10\sqrt{d}, 20\sqrt{d}]$. let φ be defined in Lemma 5.5.5. and

$E_c \overset{\text{def}}{=} \{ \overrightarrow{xy} : \varphi(y) \equiv \varphi(x) + c \mod p \}$ i.e. 

. Note $E_G = E_0 + E_1 + \cdots + E_{p-1}$.

let $G_c = (V_G, E_c)$ be a spanning subdigraph of G, for $c = 0, 1, 2, \cdots, p-1$.

Step 1: Consider the upper bounds for $dla(G_1)$, $dla(G_2)$, $\cdots$ $dla(G_{p-1})$.

max. out (in) degree of G_c Δ_c^+ (Δ_c^-) $\le \frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d}$ ($\because$ Lemma 5.5.5) $(0 < c \le p-1)$

Suppose $t = \text{digirth}(G_c)$ i.e. $\varphi(x_1) \xrightarrow{+c} \varphi(x_2) \xrightarrow{+c} \cdots\cdots \xrightarrow{+c} \varphi(x_t) \xrightarrow{+c} \varphi(x_1)$ (mod p) and hence $p \mid ct$

and so $p \mid t$. Therefore $\text{digirth}(G) \ge p$.

(here we use the property that p is a prime)

Pf (continued) Therefore there exists a Δ_c-regular digraph H s.t.

① G_c is a subdigraph of H,

② $\Delta_c = \max\{\Delta_c^+, \Delta_c^-\} \leq \frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d}$,

③ $\text{digirth}(H) = \text{digirth}(G_c) \geq p \geq 8e\{4\frac{d}{p}\} \geq 8e\Delta_c$.

$(\because p \geq 10\sqrt{d} \Rightarrow p \geq \frac{100d}{p} \geq 8\times 3\{4\frac{d}{p}\})$ $(\because d >> 1)$

Thm 5.5.4 $\Rightarrow$ $dla(G_c) \leq dla(H) \leq \Delta_c + 1 \leq \frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d} + 1$.

So

$$dla(G) \leq dla(G_0) + \underbrace{dla(G_1) + \cdots + dla(G_{p-1})}_{\leq\, (p-1)\left\{\frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d} + 1\right\}}.$$

Pf (continued)

Step 2: Consider the upper bound for $\mathrm{dla}(G_0)$. We have

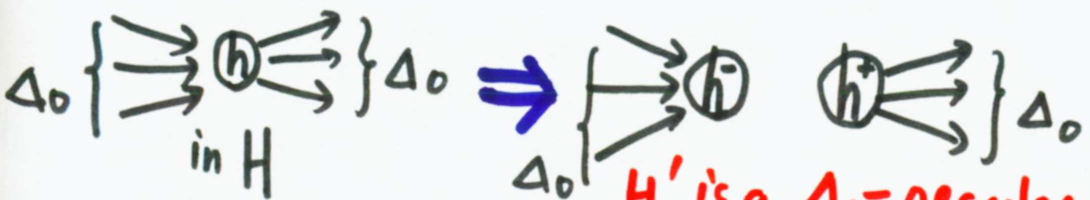
max. $\begin{matrix}\text{in}\\ \text{out}\end{matrix}$ > degree of G_0 $\begin{matrix}\Delta_0^-\\ \Delta_0^+\end{matrix} \leq \frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d}$.

Therefore $\exists$ a Δ_0-regular digraph H s.t.

① G_0 is a subdigraph of H.

② $\Delta_0 = \max\{\Delta_0^+, \Delta_0^-\} \leq \frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d}$

(note: here we can not control digirth(G_0))



$\Rightarrow$ H' has Δ_0 1-factors

H' is a Δ_0-regular bipartite graph

$\Rightarrow$ H has Δ_0 2-factors $\Rightarrow$ $\mathrm{dla}(G_0) \leq \mathrm{dla}(H) \leq 2\Delta_0 \leq 2\frac{d}{p} + 6\sqrt{\frac{d}{p}}\sqrt{\ln d}$

(decompose each 2-factor into 2 linear diforest)

pf (continued)

$$dla(G) \leq dla(G_0) + dla(G_1) + dla(G_2) + \cdots + dla(G_{p-1})$$

$$\leq 2\frac{d}{p} + 6\sqrt{\frac{d}{p}}\sqrt{\ln d} + (p-1)\left\{\frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\ln d} + 1\right\}$$

why?

$$\leq d + c d^{\frac{3}{4}} (\ln d)^{\frac{1}{2}}$$ for some constant by usig the fact that $p \in [10\sqrt{d},\ 20\sqrt{d}]$ and d sufficiently large.

QED