Birkhoff's representation theorem for finite distributive lattices

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Birkhoff, G.
"On the structure of abstract algebras"
Proc. Cambridge Phil. Soc,
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Birkhoff showed (amazingly) that partial orders are in 1-1 correspondence with distributive lattices. In particular, a distributive lattice can be constructed from the partial order of its join irreducibles. Since the construction is informative, the construction is explained by example in Figure **3 and 4.**

Garrett Birkhoff. *Lattice Theory*. American Mathematical Society, 3rd edition edition, 1967. 1st edition 1940.

Warm up

Lemma A Let x be a nonzero element of a finite distributive lattice L. Then

x is join-irreducible $\Longleftrightarrow$ for all $I \subseteq L$, $x \leq \bigvee I$ yields $x \leq y$ for some $y \in I$

pf: ($\Rightarrow$) By induction on $|I|$. It suffices to consider $I = \{z, y\}$.

$x \leq y \vee z \Rightarrow x = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$\Rightarrow x = x \wedge y$ or $x = x \wedge z$ ($\because$ joint-irreducible)

$\Rightarrow x \leq y$ or $x \leq z$ done!

($\Leftarrow$) Suppose $x = a \vee b$.

$x = a \vee b \Rightarrow x \leq a \vee b \Rightarrow x \leq a$ or $x \leq b$ (hypothesis) **QED**

$\Rightarrow x = a$ or $x = b$ ($\because x = a \vee b \geq a$, $\geq b$)

$x = \bigvee (x] \cap J(L)$

Lemma B Let L be a finite lattice and $x \in L$.
Then $x = \bigvee (x] \cap J(L)$.

pf: Let $y = \text{RHS}$ i.e. y is the least upper bound for $(x] \cap J(L)$.
Note that $y \le x$, since x is an upper bound for $(x] \cap J(L)$.
Assume $y \ne x$. Consider the set $A = (x] \cap \{a : y \ngeq a\}$.
Note that $A \ne \phi$, since $x \in A$. Let a be a minimal element of A (such a exist since A contains no infinite descending chain).
If $a = b \vee c$ then $a = b$ or $a = c$. ($\because$ otherwise $a > b$ & $a > c \Rightarrow b, c \notin A$ $\Rightarrow y \ge b$ & $y \ge c$ $\Rightarrow y \ge a$ a contradiction)
So $a \in (x] \cap J(L)$ and hence $y \ge a$ ($\because y = \bigvee (x] \cap J(L)$)
a contradiction. Therefore $y = x$.

QED

Birkhoff's representation theorem

Thm let $(L, \leq)$ be a finite distributive lattice. Then the map $\varphi(x) = (x] \cap J(L)$ is an isomorphism between $(L, \leq)$ and $(D(J(L)), \subseteq)$.

↑ a distributive lattice

pf: "1 to 1" $\varphi(x) = \varphi(y) \Rightarrow x = \bigvee (x] \cap J(L) = \bigvee (y] \cap J(L) = y$.

"onto" For $A \in D(J(L))$, let $a = \bigvee A$. clearly $\varphi(a) = (a] \cap J(L) \supseteq A$

$(\because y \in A \Rightarrow y \in J(L)$ and $y \leq a)$. On the other hand, consider $x \in \varphi(a)$.

$x \leq a \Rightarrow x = x \wedge a = x \wedge \bigvee A = \bigvee \{x \wedge y \mid y \in A\}$

$\Rightarrow x = x \wedge y$ for some $y \in A$ ($\because x$ is join-irreducible)

$\Rightarrow x \leq y$ for some $y \in A \Rightarrow x \in A$ ($\because A$ is a down-set)

Therefore $\varphi(a) \subseteq A$ and hence $\varphi(a) = A$.

pf (continued)

To show that φ is order-preserving!

$x \leq y$ in $(L, \leq)$ $\Rightarrow$ $(x] \cap J(L) \subseteq (y] \cap J(L)$

$\Rightarrow$ $\varphi(x) \subseteq \varphi(y)$ in $(D(J(L)), \subseteq)$.

$\varphi(x) \subseteq \varphi(y)$ in $(D(J(L)), \subseteq)$

$\Rightarrow$ $x = \bigvee (x] \cap J(L) \leq \bigvee (y] \cap J(L) = y$ in $(L, \leq)$.

QED

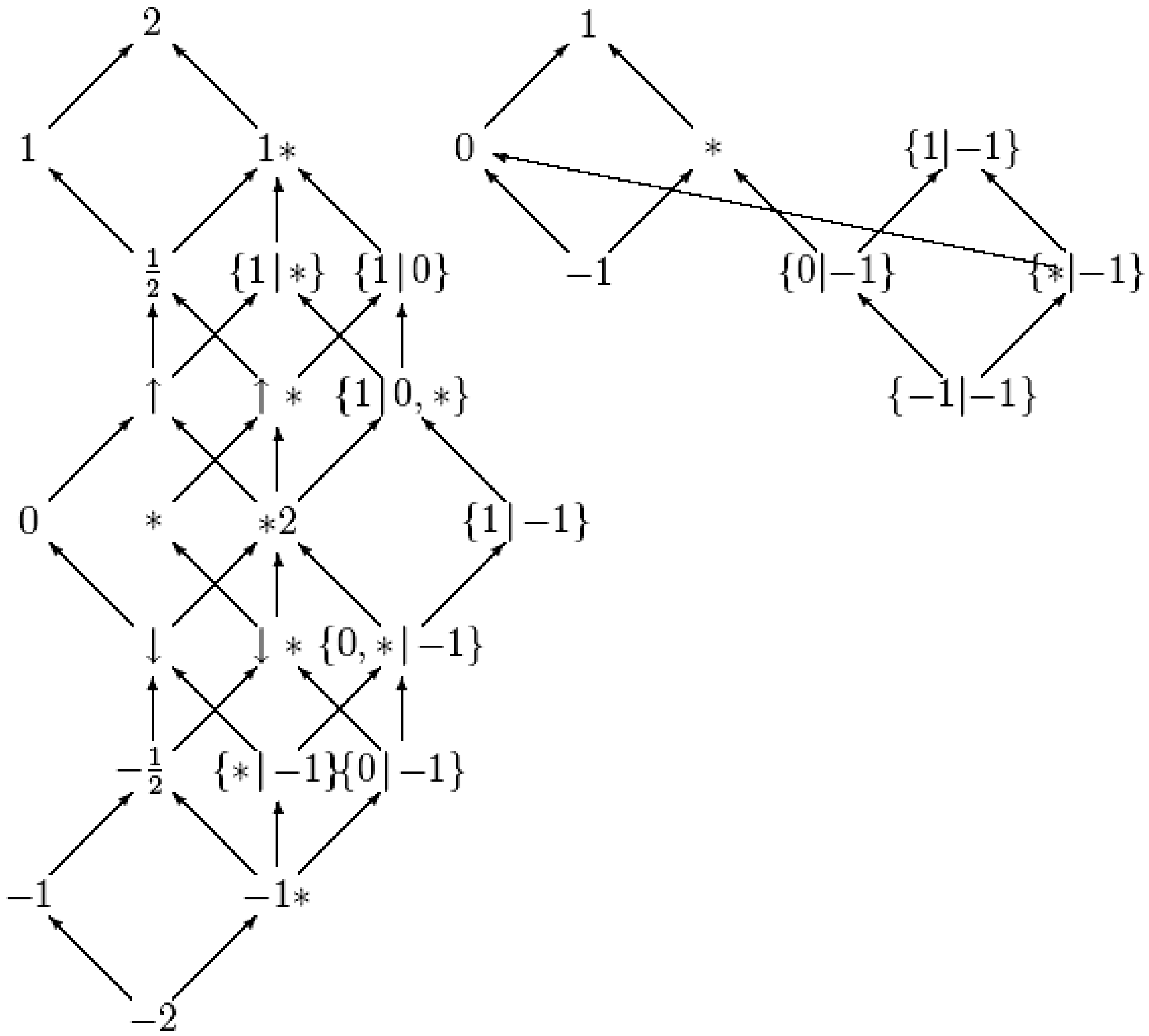
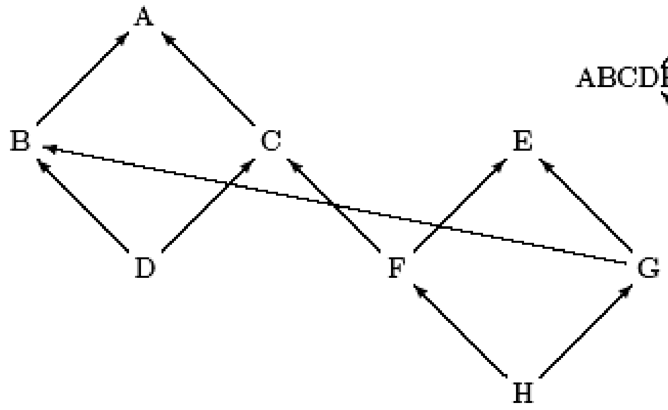


Figure 3: The day 2 lattice is shown on the right. The join irreducibles from day 2 are shown on the left.



ABCDEFGH	DFGH
ABCDFGH	EFGH
BCDEFGH	DFH
BCDFGH	DGH
BDEFGH	FGH
CDEFGH	DH
BDFGH	FH
CDFGH	GH
DEFGH	D
BDGH	H
CDFH	-

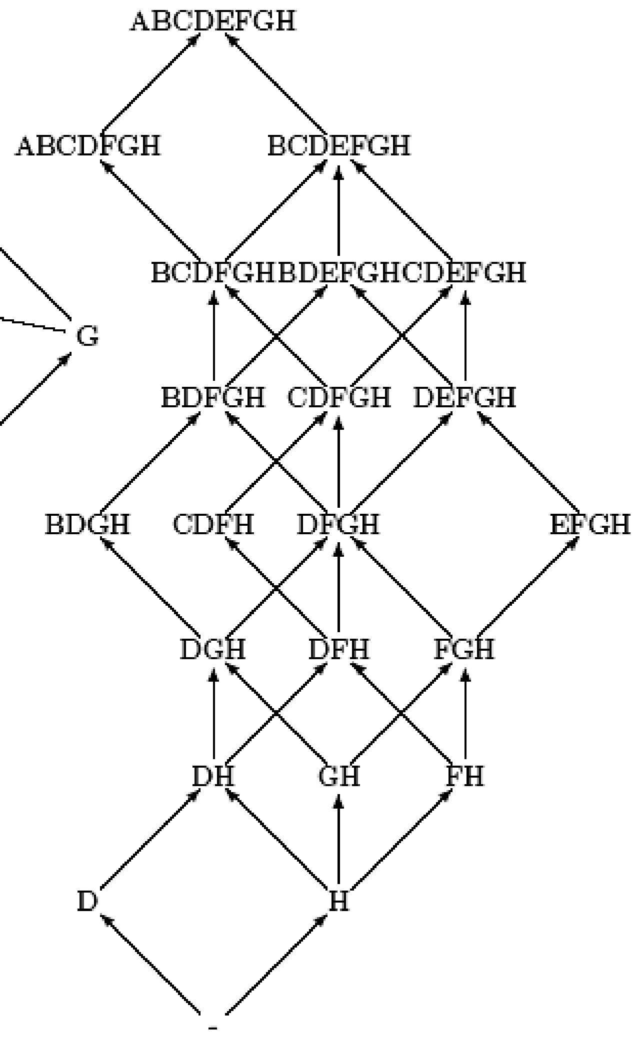
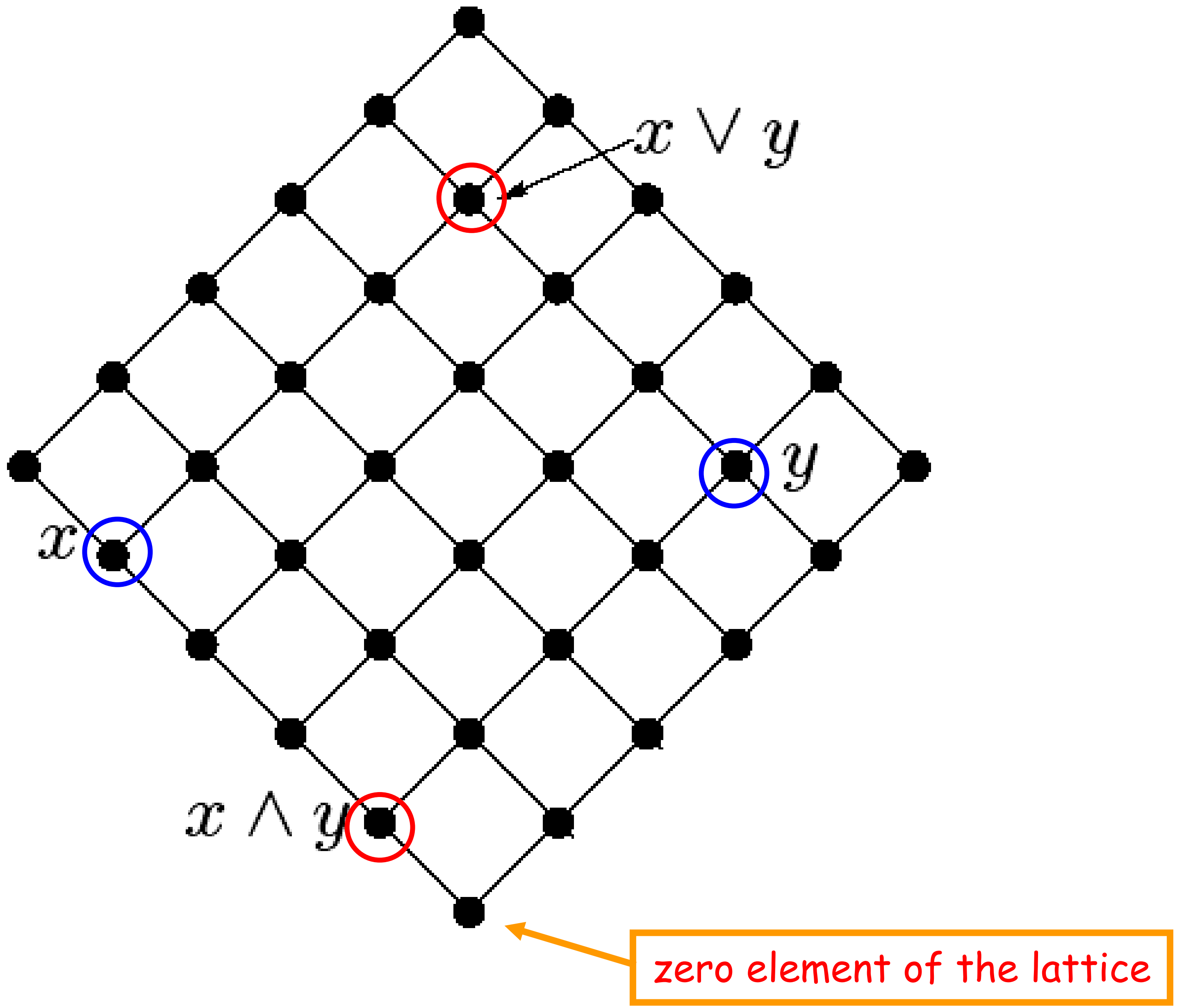


Figure 4: The upper left shows the partial order of the join-irreducible elements of the day 2 distributive lattice. The lower-left lists all 22 downsets of the partial order. A downset is a subsets, S, of elements such that if $x \leq y$ and $y \in S$ then $x \in S$. On the right is the original lattice as reconstructed from the downsets. Downset $S_1 \leq S_2$ if $S_1 \subset S_2$.


$x \vee y$
y
x
$x \wedge y$
zero element of the lattice