Partially Ordered Sets

A poset (X, P): X is a ground set. P (called a partial order) is a reflexive, antisymmetric, and transitive binary relation on X.

i.e.
1. Reflexive: $(a,a) \in P$ for $\forall\, a \in X$.
2. antisymmetric: $(a,b), (b,a) \in P \Rightarrow a = b$
3. transitive: $(a,b), (b,c) \in P \Rightarrow (a,c) \in P$

Remark: When $(x,y) \in P$, we write $x \leq y$ in P.

x and y are comparable: if either $x \leq y$ or $y \leq x$ in P.
$x \in X$, $y \in X$; else they are incomparable.

Notation: $x < y$ in P means of $x \leq y$ in P and $x \neq y$.

y covers x: i.e. ① $x < y$ and ② $\nexists\, z$ s.t. $x < z < y$ in P.

Basic Terminology (I)

point x is maximal $\equiv$ $\nexists\, y$ s.t. $x < y$ in P.

point x is minimal $\equiv$ $\nexists\, y$ s.t. $y < x$ in P.

components : A poset is naturally partitioned into components.

chain : $C \subseteq X$ is called a chain if $\forall$ pair of points in C is a comparable pair.

total order $\equiv$ a poset which is a chain.

maximal chain $\equiv$ no chain C' containing C as a proper subset.

maximum chain $\equiv$ no chain in P has more points than C.

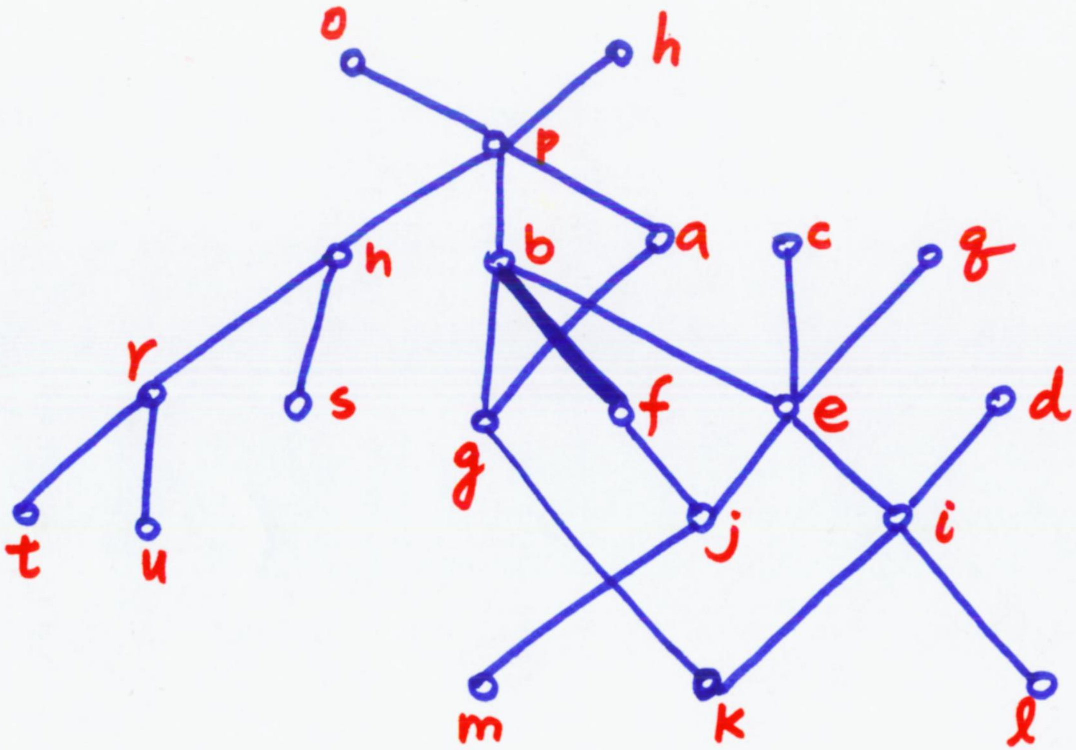
height of a poset $\equiv$ the # of points in a maximum chain.

antichain : $S \subseteq X$ is called an antichain if $\forall$ pair of points in S is an incomparable pair.

width of a poset $\equiv$ the # of points in a maximum antichain.

Hasse diagrams

Ex A poset $P = (X, P)$ with $|X| = 21$, height $(X, P) = 6$, width $(X, P) = 8$



$\max(X, P) = \{c, q, d, h, o\}$

$\min(X, P) = \{t, u, s, m, k, l\}$

X can be partitioned into 8 chains:

$C_1 = \{t, r, n, p, o\}$, $C_5 = \{l, i, e, c\}$

$C_2 = \{u, h\}$, $C_6 = \{s\}$

$C_3 = \{m, j, f, b\}$ $C_7 = \{d\}$

$C_4 = \{k, g, a\}$ $C_8 = \{q\}$

- $\{a, b, c, d, n, q\}$ is a maximal antichain.
- $\{o, p, b, e, i, k\}$ is a marimum chain

Basic Terminology (II)

$\max(X, P)$ = all maximal points of the poset (X, P)

$\min(X, P)$ = all minimal points of the poset (X, P)

an upper bound for $S \subseteq X$ $\equiv$ a $y \in X$ s.t. $x \leq y$ for all $x \in S$.

a least upper bound for S, $\mathrm{lub}(S)$ = an upper bound y for S s.t. $y \leq y'$ for any other upper bound y' for S. Remark: $\mathrm{lub}(S)$ may not exist!

Remark: a lower bound for S, a greatest lower bound for S, $\mathrm{glb}(S)$.

$x \vee y = \mathrm{lub}\{x, y\}$ $\equiv$ join of x and y

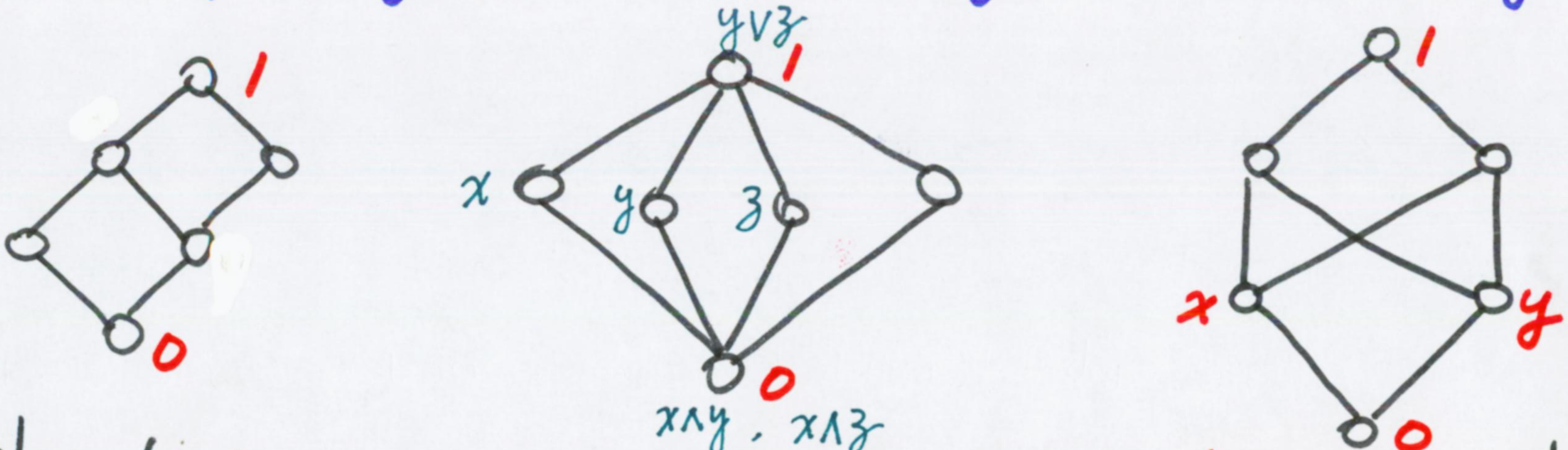
$x \wedge y = \mathrm{glb}\{x, y\}$ $\equiv$ meet of x and y

$\mathbb{X} \vee Y = \{x \vee y : x \in \mathbb{X}, y \in Y\}$ for any $\mathbb{X}, Y \subseteq X$

$\mathbb{X} \wedge Y = \{x \wedge y : x \in \mathbb{X}, y \in Y\}$ for any $\mathbb{X}, Y \subseteq X$

Lattices

Def: A poset (X, P) is a lattice if $lub(S)$ and $glb(S)$ exist for any finite nonempty subset S of X.



The first two posets (poset digrams) are lattices. The last one is not a lattice.

distributive lattice $\equiv$ a lattice (X, P) is distributive if for all $x, y, z \in X$ we have $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Basic Terminology (III)

Let (X, P) be a poset and $(L, \leq)$ be a lattice.

- $x \in L$ is **join-irreducible** if $x = a \vee b$ yields $x = a$ or $x = b$.

  $J(L)$ $\stackrel{\text{def}}{=}$ the set of all nonzero join-irreducible elements of L, $(J(L), \leq)$ is regarded as a poset.

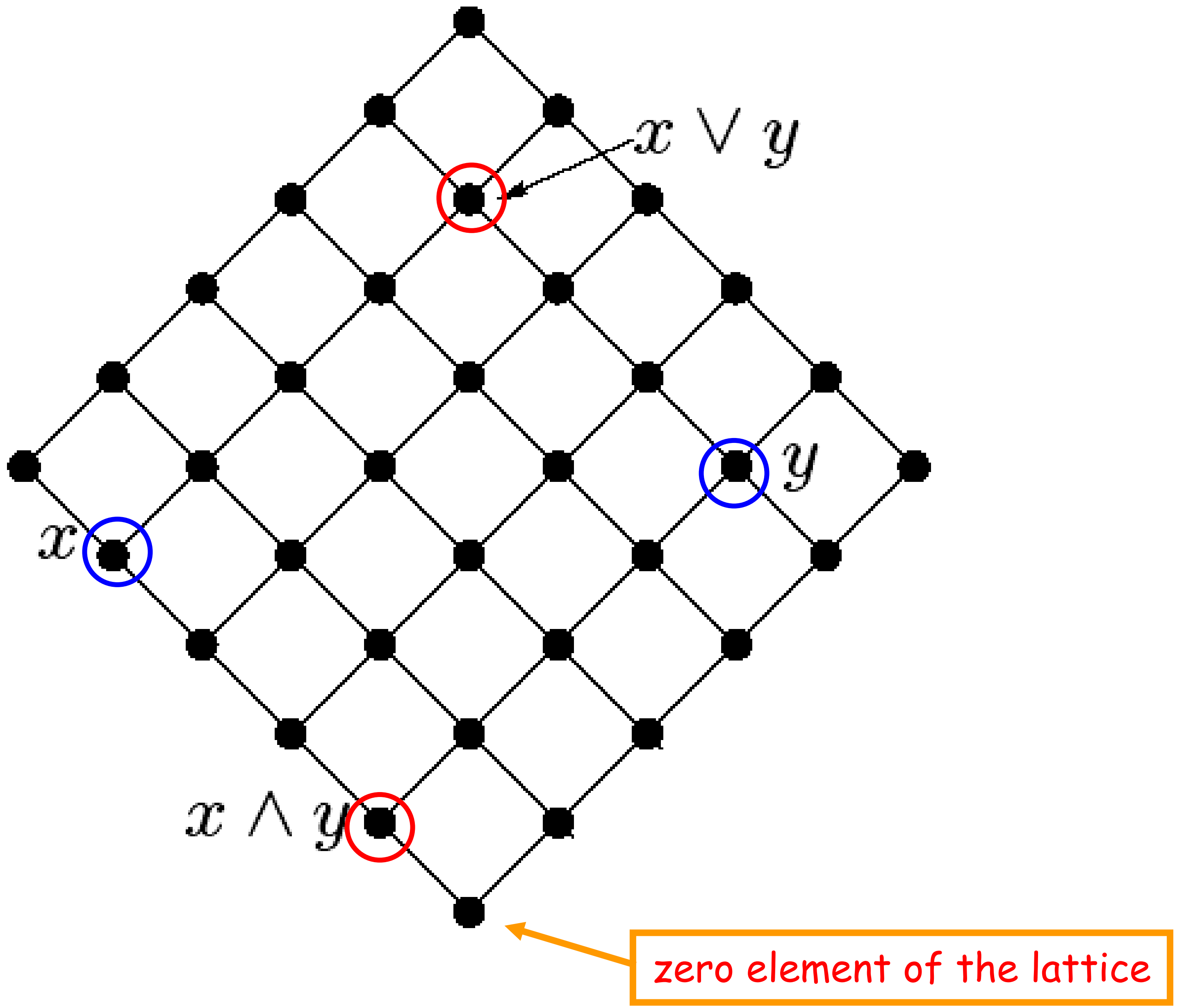
- $S \subseteq X$ is called a **down-set** in (X, P) if $\underset{\substack{\Downarrow \\ X}}{x} \leq y \in S$ yields $x \in S$.

- **$\bigvee I$** $\stackrel{\text{def}}{=}$ the join of all elements in $I \subseteq L$ under the partial ordering of L.

- **$(x] \cap J(L)$** $\stackrel{\text{def}}{=}$ $\{ y \in J(L) : y \leq x \}$

- **$D(J(L))$** $\stackrel{\text{def}}{=}$ the set of all down-set in $(J(L), \leq)$.

- Two posets X, Y are **isomorphic** if $\exists$ a bijection $\varphi : X \to Y$ s.t. $a \leq b$ in $X \iff \varphi(a) \leq \varphi(b)$ in Y.


$x \vee y$
y
x
$x \wedge y$
zero element of the lattice