The four functions theorem (FFT)

Thm Suppose $\alpha, \beta, \gamma, \delta : 2^{[n]} \longrightarrow \mathbb{R}_{\geq 0}$.

If $A, B \in 2^{[n]} \Rightarrow \alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B)$, ------(*)

then $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]} \Rightarrow \alpha(\mathcal{A})\beta(\mathcal{B}) \leq \gamma(\mathcal{A} \cup \mathcal{B})\delta(\mathcal{A} \cap \mathcal{B})$,

where $\mathcal{A} \underset{\cap}{\cup} \mathcal{B} = \{ A \underset{\cap}{\cup} B : A \in \mathcal{A}, B \in \mathcal{B}\}$ and $\alpha(\mathcal{A}) = \sum_{A \in \mathcal{A}} \alpha(A)$.

Pf: Suppose (*) holds, it suffices to show $\alpha(2^{[n]})\beta(2^{[n]}) \leq \gamma(2^{[n]})\delta(2^{[n]})$. why[1]

To prove this by induction on n.

[Basis] $n=1 \Rightarrow 2^{[n]} = \{\emptyset, [1]\}$. To prove that $(\underset{\alpha(\emptyset)}{\underset{\shortparallel}{\alpha_0}} + \underset{\alpha([1])}{\underset{\shortparallel}{\alpha_1}})(\beta_0+\beta_1) \leq (\gamma_0+\gamma_1)(\delta_0+\delta_1)$. (✱)

- $\gamma_1 = 0$ or $\delta_0 = 0 \Rightarrow$ (✱) holds. Why[2]
- $\gamma_1 \neq 0$ & $\delta_0 \neq 0 \Rightarrow \delta_1 \geq \dfrac{\alpha_1\beta_1}{\gamma_1}$ & $\gamma_0 \geq \dfrac{\alpha_0\beta_0}{\delta_0}$ by (*), and hence

$$(\gamma_1+\gamma_0)(\delta_0+\delta_1) \geq \left(\frac{\alpha_0\beta_0}{\delta_0}+\gamma_1\right)\left(\delta_0+\frac{\alpha_1\beta_1}{\gamma_1}\right) \underset{\text{expand}}{=} \alpha_0\beta_0 + \frac{\alpha_0\beta_0\alpha_1\beta_1}{\delta_0\gamma_1} + \delta_0\gamma_1 + \alpha_1\beta_1$$

$$= (\alpha_0+\alpha_1)(\beta_0+\beta_1) + \gamma_1\delta_0\left(1-\frac{\alpha_1\beta_0}{\gamma_1\delta_0}\right) - \alpha_0\beta_1\left(1-\frac{\alpha_1\beta_0}{\gamma_1\delta_0}\right) = (\alpha_0+\alpha_1)(\beta_0+\beta_1) + (\gamma_1\delta_0-\alpha_0\beta_1)\left(1-\frac{\alpha_1\beta_0}{\gamma_1\delta_0}\right)$$

$$\geq (\alpha_0+\alpha_1)(\beta_0+\beta_1).$$

pf (continued) [induction part] Suppose the thm is true for n-1.

For $f \in \{\alpha, \beta, \gamma, \delta\}$, define $f' : 2^{[n-1]} \longrightarrow \mathbb{R}_{\geq 0}$ s.t. $f'(A) = f(A) + f(A+n)$.

For any $A, B \in 2^{[n-1]}$, define $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta} : 2^{[1]} \longrightarrow \mathbb{R}_{\geq 0}$ such that

$\bar{\alpha}_0 = \alpha(A)$, $\bar{\beta}_0 = \beta(B)$, $\bar{\gamma}_0 = \gamma(A \cup B)$ and $\bar{\delta}_0 = \delta(A \cap B)$.

$\bar{\alpha}_1 = \alpha(A+n)$, $\bar{\beta}_1 = \beta(B+n)$, $\bar{\gamma}_1 = \gamma((A \cup B)+n)$ and $\bar{\delta}_1 = \delta((A \cap B)\underbrace{+n}_{\cup\{n\}})$.

can be checked ! why?

Note that, for $\bar{A}, \bar{B} \in 2^{[1]}$, $\bar{\alpha}(\bar{A})\bar{\beta}(\bar{B}) \leq \bar{\gamma}(\bar{A} \cup \bar{B})\bar{\delta}(\bar{A} \cap \bar{B})$. Therefore,

by [Basis], $\bar{\alpha}(2^{[1]})\bar{\beta}(2^{[1]}) \leq \bar{\gamma}(2^{[1]})\bar{\delta}(2^{[1]})$, that is

$$[\alpha(A) + \alpha(A+n)][\beta(B) + \beta(B+n)] \leq [\gamma(A \cup B) + \gamma((A \cup B)+n)][\delta(A \cap B) + \delta((A \cap B)+n)]$$

we have $\alpha'(A)\ \beta'(B) \leq \gamma'(A \cup B)\ \delta'(A \cap B)$.

By induction hypothesis we have $\alpha'(2^{[n-1]})\beta'(2^{[n-1]}) \leq \gamma'(2^{[n-1]})\delta(2^{[n-1]})$,

that is $\alpha(2^{[n]})\beta(2^{[n]}) \leq \gamma(2^{[n]})\delta(2^{[n]})$.

QED

Let me explain why.

Why^1: For $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$, define $\alpha^*, \beta^*, \gamma^*, \delta^* : 2^{[n]} \to \mathbb{R}_{\geq 0}$ s.t.

$$\alpha^*(S) = \alpha(S) I_{\{S \in \mathcal{A}\}}, \quad \beta^*(S) = \beta(S) I_{\{S \in \mathcal{B}\}}$$

$$\gamma^*(S) = \gamma(S) I_{\{S \in \mathcal{A} \cup \mathcal{B}\}}, \quad \delta^*(S) = \delta(S) I\{S \in \mathcal{A} \cap \mathcal{B}\}$$

Then $\alpha^*, \beta^*, \gamma^*, \delta^*$ satisfy $(*)$. Now if we can prove that $\alpha^*(2^{[n]}) \beta^*(2^{[n]}) \leq \gamma^*(2^{[n]}) \delta(2^{[n]})$. Then we get what we want $\alpha(\mathcal{A}) \beta(\mathcal{B}) \leq \gamma(\mathcal{A} \cup \mathcal{B}) \delta(\mathcal{A} \cap \mathcal{B})$.

Why^2: When $n=1$, $(*)$ implies

$$\alpha_0 \beta_0 \leq \gamma_0 \delta_0, \quad \alpha_0 \beta_1 \leq \gamma_1 \delta_0, \quad \alpha_1 \beta_0 \leq \gamma_1 \delta_0, \text{ and } \alpha_1 \beta_1 \leq \gamma_1 \delta_1.$$

Why^3: (sketch) $\alpha(A) \beta(B+n) \leq \gamma((A \cup B)+n) \delta(A \cap B)$

Extend FKG to any finite distributive lattice

Corollary: Let $(L,\leq)$ be a finite distributive lattice and let $\alpha, \beta, \gamma, \delta : L \longrightarrow \mathbb{R}_{\geq 0}$.

If $\alpha(x)\beta(y) \leq \gamma(x \vee y)\delta(x \wedge y)$ for all $x, y \in L$,

then $\alpha(X)\beta(Y) \leq \gamma(X \vee Y)\delta(X \wedge Y)$ for all $X, Y \subseteq L$.

pf: $n \overset{\text{def}}{=} |J(L)|$, $J_x \overset{\text{def}}{=} (x] \cap J(L)$, $\mathcal{J}_X \overset{\text{def}}{=} \{J_x : x \in X\}$ for any $X \subseteq L$.

Birkhoff's representation thm says that $(L,\leq) \cong (D(J(L)), \subseteq)$ with an isomorphism $\varphi(x) = J_x$. Now for $f \in \{\alpha, \beta, \gamma, \delta\}$, define $\overline{f} : 2^{J(L)} \longrightarrow \mathbb{R}_{\geq 0}$ s.t.
$\overline{f}(A) = \begin{cases} f(x) & \text{if } A = J_x \text{ for some } x \in L \\ 0 & \text{o.w.} \end{cases}$. For any $A, B \in 2^{J(L)}$, we have
$\overline{\alpha}(A)\overline{\beta}(B) \leq \overline{\gamma}(A \cup B)\overline{\delta}(A \cap B)$ (this follows from the hypothesis of this corollary and the facts that if $A = J_x$ and $B = J_y$ then $A \cup B = J_{x \vee y}$, $A \cap B = J_{x \wedge y}$ why? prove it!)

b

pf (continued)

Therefore, by four functions theorem, we arrive at

for any $X, Y \subseteq L$ we have $\bar{\alpha}(\mathcal{J}_X)\,\bar{\beta}(\mathcal{J}_Y) \leq \bar{\gamma}(\mathcal{J}_X \cup \mathcal{J}_Y)\,\bar{\delta}(\mathcal{J}_X \cap \mathcal{J}_Y)$

that is $\alpha(X)\beta(Y) \leq \gamma(X \cup Y)\,\delta(X \cap Y)$.

QED