The FKG inequality (I)

log-supermodular: $(L, \leq)$ is a finite distributive lattice (f.d. lattice).

$\mu : L \to \mathbb{R}_{\geq 0}$ is log-supermodular if $\mu(x)\mu(y) \leq \mu(x \vee y)\mu(x \wedge y)$.

increasing (decreasing): $f : L \to \mathbb{R}_{\geq 0}$ is $\uparrow (\downarrow)$ if $x \leq y \Rightarrow f(x) \underset{(\geq)}{\leq} f(y)$

Thm If $(L, \leq)$ is a f.d. lattice & $\mu : L \to \mathbb{R}_{\geq 0}$ is log-supermodular, then for any two $\uparrow$ functions $f, g : L \to \mathbb{R}_{\geq 0}$ we have

$$\left(\sum_{x\in L} \mu(x) f(x)\right)\left(\sum_{x\in L} \mu(x) g(x)\right) \leq \left(\sum_{x\in L} \mu(x) f(x) g(x)\right)\left(\sum_{x\in L} \mu(x)\right)$$

pf: Define $\alpha, \beta, \gamma, \delta : L \to \mathbb{R}_{\geq 0}$ s.t. $\alpha = \mu f$, $\beta = \mu g$, $\gamma = \mu f g$, $\delta = \mu$.

For $x, y \in L$, we have

$$\alpha(x)\beta(y) = \mu(x) f(x) \mu(y) g(y) \overset{(\because \text{log-supermodular})}{\leq} \underline{\mu(x\vee y)}\, \mu(x \wedge y)\, \underline{f(x)\, g(y)} \overset{(\because f, g \uparrow)}{\leq} \gamma(x \vee y)\, \delta(x \wedge y).$$

FKT $\Longrightarrow$ for $\forall X, Y \subseteq L$, $\alpha(X)\beta(Y) \leq \gamma(X \vee Y)\,\delta(X \wedge Y)$

$\Longrightarrow \alpha(L)\beta(L) \leq \gamma(L)\delta(L)$ done!

QED

The FKG inequality (II)

Corollary 1 If $f\downarrow$ & $g\downarrow$ on L, then FKG inequality also holds.

pf: In the proof of FKG ineq., let $\alpha=\mu f$, $\beta=\mu g$, $\gamma=\mu$ & $\delta=\mu f g$ to get $\alpha(x)\beta(x)\underset{\text{log-supermodular}}{\le} \underline{\mu(x\vee y) f(x)\mu(x\wedge y) g(x)} \underset{f,g\downarrow}{\le} \mu(x\vee y)\,\delta(x\wedge y)$.

QED

Corollary 2 If $f\uparrow$ & $g\downarrow$ (or vice versa), then in FKG inequality, the oppositive inequality holds.

pf: let $k=\max_{x\in L} g(x)$. Then we have $f\uparrow$ & $k-g\uparrow$ from L to $\mathbb{R}_{\ge 0}$.

FKG ineq. $\Rightarrow [\mu f(L)][\mu(k-g)(L)] \le [\mu f(k-g)(L)][\mu(L)]$

$\Rightarrow [\mu f][\mu k-\mu g] \le [\mu f k-\mu f g]\,\mu$

omit L

$\Rightarrow \underbrace{[\mu f][\mu k]-[\mu f k][\mu]}_{0} \le [\mu f][\mu g]-[\mu f g]\,\mu$

$\Rightarrow [(\mu f)(L)]\,[(\mu g)(L)] \ge [(\mu f g)(L)]\,[\mu(L)]$.

QED

View μ as a measure on L

Thm If L is a finite distributive lattice and $\mu : L \to \mathbb{R}_{\geq 0}$ is a log-supermodular function which is not a zero function. Then for any $f\uparrow$ & $g\uparrow : L \to \mathbb{R}_{\geq 0}$, we have

$$\mathcal{E}(fg) \geq (\mathcal{E}f)(\mathcal{E}g)$$

where $\mathcal{E}f = \dfrac{\sum_{x\in L} f(x)\mu(x)}{\mu(L)}$.

Remark: This thm demonstrates the probabilistic nature of the FKG ineq.