The Harris- Kleitman inequality

1960 1966

- $\mathcal{A} \in 2^{\Omega}$ (event) is a downset (upset) if $B \subset A \in \mathcal{A}$ $(B \supset A \in \mathcal{A}) \Rightarrow B \in \mathcal{A}$.

  $\mathcal{A}\downarrow$ $\mathcal{A}\uparrow$
- $(\Omega, 2^{\Omega}, \mathcal{P})$ is a prob. space s.t. $\mathcal{P}(\{A\}) = \prod_{i\in A} p_i \prod_{j \in \bar{A}} (1-p_j)$, where $0 \le p_i \le 1$

  sample space

  and $\Omega = 2^{[n]}$ (w.l.o.g).

Thm For any $0 \le p_i \le 1$, $i = 1, 2, \cdots, n$, and $\mathcal{A}, \mathcal{B} \subseteq \Omega$.

$\mathcal{A}\uparrow$ and $\mathcal{B}\uparrow$ $\Rightarrow$ $\mathcal{P}(\mathcal{A} \cap \mathcal{B}) \ge \mathcal{P}(\mathcal{A})\mathcal{P}(\mathcal{B})$ positively correlated

usual intersection

$\uparrow$ upset in 2^{Ω} $\downarrow$ $\le$ negatively

$\downarrow$ downset in 2^{Ω} $\downarrow$ $\ge$ positively

pf: $(\Omega, \subseteq)$ is a finite distributive lattice. $\mu(A) \overset{\text{def}}{=\!=} \mathcal{P}(\{A\})$ $\forall A \in \Omega$.

Let $I_{\mathcal{A}}$, $I_{\mathcal{B}}$ be two indicate funs. on Ω.

Claim: μ is log-supermodular. pf: By the fact that $(\prod_{i\in A} p_i)(\prod_{i\in B} p_i) = (\prod_{i \in A\cup B} p_i)(\prod_{i\in A\cap B} p_i)$.

Claim: $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ are $\uparrow$ in the lattice $(\Omega, \subseteq)$. pf: By the fact that $\mathcal{A}\uparrow$ and $\mathcal{B}\uparrow$.

$$\mathcal{P}(\mathcal{A} \cap \mathcal{B}) = \left(\sum_{A\in\Omega} \mu(A) I_{\mathcal{A}}(A) I_{\mathcal{B}}(A)\right)\left(\sum_{A\in\Omega}\mu(A)\right)$$

$$\underset{\because \text{FKG}}{\ge} \left(\sum_{A\in\Omega}\mu(A) I_{\mathcal{A}}(A)\right)\left(\sum_{A\in\Omega}\mu(A) I_{\mathcal{B}}(A)\right) = \mathcal{P}(\mathcal{A})\mathcal{P}(\mathcal{B}).$$

QED