The Janson inequality

- $\Omega \overset{\text{def}}{=} 2^{[n]}$, $0 \leq p_i \leq 1$ for $i = 1, 2, \cdots, n$.
- $(\Omega, 2^{\Omega}, P)$ is a prob. space s.t. $P(\{\omega\}) = \prod_{i \in \omega} p_i \prod_{j \in \overline{\omega}} (1-p_j)$ for $\forall \omega \in \Omega$.
- Let $A_1, \cdots, A_m \in \Omega$ be fixed s.t. $i \sim j \Leftrightarrow A_i \cap A_j \neq \phi$.
  ordered pair
- $B_i \overset{\text{def}}{=} \{\omega \in \Omega : A_i \subseteq \omega\}$, $i = 1, 2, \cdots, m$.

Thm Suppose $P(B_i) \leq \varepsilon < 1$, $\forall i$. $\mu \overset{\text{def}}{=} \sum_{i \in [m]} P(B_i)$, $M \overset{\text{def}}{=} \prod_{i \in [m]} P(\overline{B_i})$ and $\Delta \overset{\text{def}}{=} \sum_{i \sim j} P(B_i \cap B_j)$. Then we have

(1) $M \overset{①}{\leq} P\left(\bigcap_{i=1}^{m} \overline{B_i}\right) \overset{②}{\leq} M e^{\frac{\Delta}{2(1-\varepsilon)}}$, and

(2) $P\left(\bigcap_{i=1}^{m} \overline{B_i}\right) \overset{③}{\leq} e^{-\mu + \frac{\Delta}{2}}$

pf. claimA For $i, k \notin J \subseteq [m]$, we have

ⓐ $P(\overline{B}_i \mid \wedge_{j\in J}\overline{B}_j) \geq P(\overline{B}_i)$, and

ⓑ $P(B_i \mid B_k \cap \wedge_{j\in J}\overline{B}_j) \leq P(B_i \mid B_k)$.

pf(claimA): $\omega' \subseteq \omega \in \overline{B}_i \Rightarrow \omega' \in \overline{B}_i$ and $\omega' \supseteq \omega \in B_i \Rightarrow \omega' \in B_i$.

ⓐ $\overline{B}_i$ and $\underbrace{\cap_{j\in J}\overline{B}_j}_{\mathcal{B}}$ are downsets $\Rightarrow P(\overline{B}_i \cap \mathcal{B}) \geq P(\overline{B}_i)P(\mathcal{B})$ ($\because$ H-K ineq.)

ⓑ B_i is an upset and $\mathcal{B}$ is a downset $\Rightarrow P_{p_1=p_2=\cdots=p_\ell=1}(B_i\cap\mathcal{B}) \leq P_{p_1=p_2=\cdots=p_\ell=1}(B_i)\,P_{p_1=p_2=\cdots=p_\ell=1}(\mathcal{B})$ ($\because$ H-K ineq.),

(where we assume $A_k = \{1, 2, \cdots, \ell\}$ w.l.o.g.) $\Rightarrow P(B_i\cap\mathcal{B}\mid B_k) \leq P(B_i\mid B_k)\,P(\mathcal{B}\mid B_k)$

$$\Rightarrow \frac{P(B_i\cap\mathcal{B}\cap B_k)}{P(\mathcal{B}\cap B_k)} \leq P(B_i\mid B_k) \Rightarrow P(B_i\mid B_k\cap\mathcal{B}) \leq P(B_i\mid B_k) \text{ done!}$$

QED of claim

pf(ineq ①)

$$P\left(\bigcap_{i=1}^{m}\overline{B}_i\right) = \prod_{i=1}^{m} P\left(\overline{B}_i \,\Big|\, \bigcap_{1\leq j<i}\overline{B}_j\right) \quad \left(\text{let } \bigcap_{1\leq j<1}\overline{B}_j = \Omega\right)$$

$$\geq \prod_{i=1}^{m} P(\overline{B}_i) \quad (\because \text{claim A ⓐ})$$

pf (continued)

pf (ineq②) $P\left(\bigcap_{i=1}^{m}\overline{B_i}\right) = \prod_{i=1}^{m} P\left(\overline{B_i} \,\middle|\, \bigcap_{1\le j<i}\overline{B_j}\right)$

$$= \prod_{i=1}^{m}\left\{1-P\Big(B_i\cap\overbrace{\bigwedge_{\substack{j\sim i\\ j<i}}\overline{B_j}}^{B}\cap\overbrace{\bigwedge_{\substack{j\not\sim i\\ j<i}}\overline{B_j}}^{C}\Big)\Big/P(B\cap C)\right\} \le \prod_{i=1}^{m}\left\{1-\frac{P(B_i\cap B\cap C)}{P(B_i\cap C)}\,\frac{P(B_i\cap C)}{P(C)}\right\}$$

$$= \prod_{i=1}^{m}\left\{1-P(B\mid B_i\cap C)\,P(B_i)\right\} \le \prod_{i=1}^{m}\left\{1-\Big\{1-\sum_{j\sim i,\,j<i}P(B_j\mid B_i\cap C)\Big\}P(B_i)\right\}$$

$\because P(B_i\cap C)=P(B_i)P(C)$

$$\le \prod_{i=1}^{m}\left\{1-\Big\{1-\sum_{j\sim i,\,j<i}P(B_j\mid B_i)\Big\}P(B_i)\right\} = \prod_{i=1}^{m}\left\{1-P(B_i)+\sum_{j\sim i,\,j<i}P(B_i\cap B_j)\right\} = \bigstar$$

by claim A ⓑ

$$\le \prod_{i=1}^{m} P(\overline{B_i})\Big\{1+\frac{1}{1-\varepsilon}\underbrace{\sum_{j\sim i,\,j<i}P(B_i\cap B_j)}_{\alpha_i}\Big\} \le \prod_{i=1}^{m} P(\overline{B_i})\, e^{\frac{1}{1-\varepsilon}\alpha_i} = M e^{\frac{1}{1-\varepsilon}\sum_{i=1}^{m}\alpha_i} = M e^{\frac{\Delta}{2(1-\varepsilon)}}.$$

pf (ineq③) $P\left(\bigcap_{i=1}^{m}\overline{B_i}\right) \le \bigstar$

$$\le \prod_{i=1}^{m}\exp\left\{-P(B_i)+\alpha_i\right\}$$

$$= \exp\left\{-\mu+\sum_{i=1}^{m}\alpha_i\right\} = e^{-\mu+\frac{\Delta}{2}}.$$

QED