The FKG inequality (I)

<u>log-supermodular</u>: $(L, \leq)$ is a finite distributive lattice (f.d. lattice).

$\mu : L \to \mathbb{R}_{\geq 0}$ is log-supermodular if $\mu(x)\mu(y) \leq \mu(x \vee y)\mu(x \wedge y)$.

<u>increasing (decreasing)</u>: $f : L \to \mathbb{R}_{\geq 0}$ is $\uparrow$ ($\downarrow$) if $x \leq y \Rightarrow f(x) \underset{(\geq)}{\leq} f(y)$

Thm If $(L, \leq)$ is a f.d. lattice & $\mu : L \to \mathbb{R}_{\geq 0}$ is log-supermodular, then for any two $\uparrow$ functions $f, g : L \to \mathbb{R}_{\geq 0}$ we have

$$\left(\sum_{x\in L} \mu(x) f(x)\right)\left(\sum_{x \in L} \mu(x) g(x)\right) \leq \left(\sum_{x\in L} \mu(x) f(x) g(x)\right)\left(\sum_{x\in L} \mu(x)\right)$$

pf: Define $\alpha, \beta, \gamma, \delta : L \to \mathbb{R}_{\geq 0}$ s.t. $\alpha = \mu f$, $\beta = \mu g$, $\gamma = \mu f g$, $\delta = \mu$.

For $x, y \in L$, we have

$$\alpha(x)\beta(y) = \mu(x) f(x) \mu(y) g(y) \overset{(\because \text{log-supermodular})}{\leq} \underline{\mu(x\vee y)}\, \mu(x \wedge y)\, \underline{f(x) g(y)} \overset{(\because f, g\, \uparrow)}{\leq} \gamma(x \vee y)\, \delta(x \wedge y).$$

FKFT $\Rightarrow$ for $\forall X, Y \subseteq L$, $\alpha(X)\beta(Y) \leq \gamma(X \vee Y)\,\delta(X \wedge Y)$

$\Rightarrow \alpha(L)\beta(L) \leq \gamma(L)\delta(L)$ done!

QED

The FKG inequality (II)

Corollary 1 If $f\downarrow$ & $g\downarrow$ on L, then FKG inequality also holds.

pf: In the proof of FKG ineq., let $\alpha=\mu f$, $\beta=\mu g$, $\gamma=\mu$ & $\delta=\mu f g$

to get $\alpha(x)\beta(x) \underset{\text{log-supermodular}}{\le} \underbrace{\mu(x\vee y) f(x)\mu(x\wedge y) g(x)}_{} \underset{f,g\downarrow}{\le} \mu(x\vee y)\,\delta(x\wedge y)$.

QED

Corollary 2 If $f\uparrow$ & $g\downarrow$ (or vice versa), then in FKG inequality, the oppositive inequality holds.

pf: Let $k=\max\limits_{x\in L} g(x)$. Then we have $f\uparrow$ & $k-g\uparrow$ from L to $\mathbb{R}_{\ge 0}$.

FKG ineq. $\Rightarrow [\mu f(L)][\mu(k-g)(L)] \le [\mu f(k-g)(L)][\mu(L)]$

$\Rightarrow [\mu f][\mu k-\mu g] \le [\mu f k-\mu f g]\,\mu$

$\Rightarrow \underbrace{[\mu f][\mu k] - [\mu f k][\mu]}_{0} \le [\mu f][\mu g] - [\mu f g]\,\mu$ (omit L)

$\Rightarrow [(\mu f)(L)][(\mu g)(L)] \ge [(\mu f g)(L)][\mu(L)]$.

QED

View μ as a measure on L

Thm If L is a finite distributive lattice and $\mu : L \to \mathbb{R}_{\geq 0}$ is a log-supermodular function which is not a zero function. Then for any $f\uparrow$ & $g\uparrow : L \to \mathbb{R}_{\geq 0}$, we have

$$\mathcal{E}(fg) \geq (\mathcal{E}f)(\mathcal{E}g)$$

where $\mathcal{E}f = \dfrac{\sum_{x\in L} f(x)\mu(x)}{\mu(L)}$.

Remark: This thm demonstrates the probabilistic nature of the FKG ineq.

The Harris-Kleitman inequality

1960 1966

- event: $\mathcal{A} \in 2^{\Omega}$ is a downset (upset) if $B \subset A \in \mathcal{A}$ $(B \supset A \in \mathcal{A}) \Rightarrow B \in \mathcal{A}$.
  $\mathcal{A}\downarrow$ $\mathcal{A}\uparrow$
- $(\Omega, 2^{\Omega}, \mathcal{P})$ is a prob. space s.t. $\mathcal{P}(\{A\}) = \prod_{i\in A} p_i \prod_{j \in \bar{A}} (1-p_j)$, where $0 \le p_i \le 1$
  sample space
  and $\Omega = 2^{[n]}$ (w.l.o.g.).

Thm For any $0 \le p_i \le 1$, $i = 1, 2, \dots, n$, and $\mathcal{A}, \mathcal{B} \subseteq \Omega$.

$\mathcal{A}\uparrow$ and $\mathcal{B}\uparrow$ $\Rightarrow$ $\mathcal{P}(\mathcal{A} \cap \mathcal{B}) \geq \mathcal{P}(\mathcal{A})\mathcal{P}(\mathcal{B})$ positively correlated

usual intersection

$\uparrow$ upset in 2^{Ω} $\downarrow$ $\leq$ negatively

$\downarrow$ downset in 2^{Ω} $\downarrow$ $\geq$ positively

pf: $(\Omega, \subseteq)$ is a finite distributive lattice. $\mu(A) \overset{\text{def}}{=} \mathcal{P}(\{A\})$ $\forall A \in \Omega$.

Let $I_{\mathcal{A}}$, $I_{\mathcal{B}}$ be two indicate funs. on Ω.

Claim: μ is log-supermodular. pf: By the fact that $\left(\prod_{i\in A} p_i\right)\left(\prod_{i\in B} p_i\right) = \left(\prod_{i \in A\cup B} p_i\right)\left(\prod_{i\in A\cap B} p_i\right)$.

Claim: $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ are $\uparrow$ in the lattice $(\Omega, \subseteq)$. pf: By the fact that $\mathcal{A}\uparrow$ and $\mathcal{B}\uparrow$.

$$\mathcal{P}(\mathcal{A} \cap \mathcal{B}) = \left(\sum_{A\in\Omega} \mu(A) I_{\mathcal{A}}(A) I_{\mathcal{B}}(A)\right)\left(\sum_{A\in\Omega}\mu(A)\right)$$

$$\underset{\because \text{FKG}}{\geq} \left(\sum_{A\in\Omega}\mu(A) I_{\mathcal{A}}(A)\right)\left(\sum_{A\in\Omega}\mu(A) I_{\mathcal{B}}(A)\right) = \mathcal{P}(\mathcal{A})\mathcal{P}(\mathcal{B}).$$

QED

The Janson inequality

- $\Omega \overset{\text{def}}{=} 2^{[n]}$, $0 \leq p_i \leq 1$ for $i = 1, 2, \cdots, n$.
- $(\Omega, 2^{\Omega}, P)$ is a prob. space s.t. $P(\{\omega\}) = \prod_{i\in\omega} p_i \prod_{j\in\overline{\omega}}(1-p_j)$ for $\forall \omega \in \Omega$.
- Let $A_1, \cdots, A_m \subseteq \Omega$ be fixed s.t. $i \sim j \Leftrightarrow A_i \cap A_j \neq \phi$.

  ordered pair
- $B_i \overset{\text{def}}{=} \{\omega \in \Omega : A_i \subseteq \omega\}$, $i = 1, 2, \cdots, m$.

Thm Suppose $P(B_i) \leq \varepsilon < 1$, $\forall i$. $\mu \overset{\text{def}}{=} \sum_{i\in[m]} P(B_i)$, $M \overset{\text{def}}{=} \prod_{i\in[m]} P(\overline{B_i})$ and $\Delta \overset{\text{def}}{=} \sum_{i\sim j} P(B_i \cap B_j)$. Then we have

(1) $M \overset{①}{\leq} P\left(\bigcap_{i=1}^{m} \overline{B_i}\right) \overset{②}{\leq} M e^{\frac{\Delta}{2(1-\varepsilon)}}$, and

(2) $P\left(\bigcap_{i=1}^{m} \overline{B_i}\right) \overset{③}{\leq} e^{-\mu + \frac{\Delta}{2}}$

pf. claimA For $i, k \notin J \subseteq [m]$, we have

ⓐ $P(\overline{B_i} \mid \wedge_{j\in J} \overline{B_j}) \geq P(\overline{B_i})$, and

ⓑ $P(B_i \mid B_k \cap \wedge_{j\in J} \overline{B_j}) \leq P(B_i \mid B_k)$.

pf(claimA): $\omega' \subseteq \omega \in \overline{B_i} \Rightarrow \omega' \in \overline{B_i}$ and $\omega' \supseteq \omega \in B_i \Rightarrow \omega' \in B_i$.

ⓐ $\overline{B_i}$ and $\underbrace{\cap_{j\in J}\overline{B_j}}_{\mathcal{B}}$ are downsets $\Rightarrow P(\overline{B_i} \cap \mathcal{B}) \geq P(\overline{B_i}) P(\mathcal{B})$ ($\because$ H-K ineq.)

ⓑ B_i is an upset and $\mathcal{B}$ is a downset $\Rightarrow P_{p_1=p_2=\cdots=p_\ell=1}(B_i \cap \mathcal{B}) \leq P_{p_1=p_2=\cdots=p_\ell=1}(B_i) P_{p_1=p_2=\cdots=p_\ell=1}(\mathcal{B})$ ($\because$ H-K ineq),

(where we assume $B_k = \{1, 2, \cdots, \ell\}$ w.l.o.g) $\Rightarrow P(B_i \cap \mathcal{B} \mid B_k) \leq P(B_i \mid B_k) P(\mathcal{B} \mid B_k)$

$$\Rightarrow \frac{P(B_i \cap \mathcal{B} \cap B_k)}{P(\mathcal{B} \cap B_k)} \leq P(B_i \mid B_k) \Rightarrow P(B_i \mid B_k \cap \mathcal{B}) \leq P(B_i \mid B_k) \text{ done!}$$

QED of claim

pf(ineq ①) $$P\left(\bigcap_{i=1}^{m} \overline{B_i}\right) = \prod_{i=1}^{m} P\left(\overline{B_i} \mid \bigcap_{1\leq j<i} \overline{B_j}\right) \quad \left(\text{let } \bigcap_{1\leq j<1} \overline{B_j} = \Omega\right)$$

$$\geq \prod_{i=1}^{m} P(\overline{B_i}) \quad (\because \text{claim A ⓐ})$$

pf (continued) <u>pf (ineq ②)</u> $P(\bigcap_{i=1}^{m}\overline{B_i}) = \prod_{i=1}^{m} P(\overline{B_i} \mid \bigcap_{1\le j<i}\overline{B_j})$

$$= \prod_{i=1}^{m}\left\{1 - P\Big(B_i \cap \overbrace{\bigwedge_{\substack{j\sim i\\ j<i}}\overline{B_j}}^{B} \cap \overbrace{\bigwedge_{\substack{j\not\sim i\\ j<i}}\overline{B_j}}^{C}\Big)\Big/P(B\cap C)\right\} \le \prod_{i=1}^{m}\left\{1 - \frac{P(B_i\cap B\cap C)}{P(B_i\cap C)}\,\frac{P(B_i\cap C)}{P(C)}\right\}$$

$$= \prod_{i=1}^{m}\left\{1 - P(B\mid B_i\cap C)\,P(B_i)\right\} \le \prod_{i=1}^{m}\left\{1 - \Big\{1-\sum_{j\sim i,\, j<i} P(B_j\mid B_i\cap C)\Big\}P(B_i)\right\}$$

$\because P(B_i\cap C) = P(B_i)P(C)$

$$\le \prod_{i=1}^{m}\left\{1 - \Big\{1-\sum_{j\sim i,\, j<i} P(B_j\mid B_i)\Big\}P(B_i)\right\} = \prod_{i=1}^{m}\left\{1 - P(B_i) + \sum_{j\sim i,\, j<i} P(B_i\cap B_j)\right\} = \bigstar$$

by claim A ⓑ

$$\le \prod_{i=1}^{m} P(\overline{B_i})\left\{1 + \frac{1}{1-\varepsilon}\underbrace{\sum_{j\sim i,\, j<i} P(B_i\cap B_j)}_{\alpha_i}\right\} \le \prod_{i=1}^{m} P(\overline{B_i})\,e^{\alpha_i} = M e^{\frac{1}{1-\varepsilon}\sum_{i=1}^{m}\alpha_i} = M e^{\frac{\Delta}{2(1-\varepsilon)}}.$$

<u>pf (ineq ③)</u> $P(\bigcap_{i=1}^{m}\overline{B_i}) \le \bigstar$

$$\le \prod_{i=1}^{m}\exp\left\{-P(B_i) + \alpha_i\right\}$$

$$= \exp\left\{-\mu + \sum_{i=1}^{m}\alpha_i\right\} = e^{-\mu + \frac{\Delta}{2}}.$$

QED

The XYZ-Theorem (I)

Thm (Shepp 1982) Let the incomes $x_1, \dots, x_n$ of n individuals be initially ordered at random uniformly on all permutations. Suppose some partial information is available on the true ordering of the x's;

e.g. $\Gamma = \{ x_1 < x_{12},\ x_7 < x_5, \dots \}$. Then for any Γ, we have

$$P(x_1 < x_2 \mid \Gamma) \leq P(x_1 < x_2 \mid \Gamma,\ x_1 < x_3).$$

pf: $[N]^n \overset{\text{def}}{=\!=} \overbrace{[N]\times\cdots\times[N]}^{n \text{ terms}}$

If $x, y \in [N]^n$ then we write $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$.

Claim a. $([N]^n, \leq)$ is a lattice, where $x \leq y$ iff $x_1 \geq y_1$ and $x_i - x_1 \leq y_i - y_1$ for $i = 1, 2, 3, \cdots, n$.

pf of claim a: Clearly $([N]^n, \leq)$ is a poset and that

$$(x \wedge y)_i = \min(x_i - x_1, y_i - y_1) + \max(x_1, y_1). \quad i = 1, 2, \cdots, n$$

$$(x \vee y)_i = \max(x_i - x_1, y_i - y_1) + \min(x_1, y_1). \quad i = 1, 2, \cdots, n$$

Thus $([N]^n, \leq)$ is a lattice.

END

Note: we need to check that $(x\wedge y)_i, (x \vee y)_i \in [N]$! Prove it!

Exercise

pf (continued) **Claim b.** $([N]^n, \leq)$ is a distributive lattice.

pf of claim b: $(x \wedge (y \vee z))_i$

$$= \min(x_i - x_1,\ (y\vee z)_i - (y \vee z)_1) + \max(x_1,\ (y \vee z)_1)$$

$$= \min(x_i - x_1,\ \max(y_i - y_1,\ z_i - z_1)) + \max(x_1,\ \min(y_1, z_1))$$

$$= \max(\min(x_i - x_1,\ y_i - y_1),\ \min(x_i - x_1,\ z_i - z_1)) + \min(\max(x_1, y_1),\ \max(x_1, z_1))$$

$$= \max((x\wedge y)_i - (x \wedge y)_1,\ (x \wedge z)_i - (x \wedge z)_1) + \min((x \wedge y)_1,\ (x \wedge z)_1)$$

$$= ((x\wedge y) \vee (x \wedge z))_i$$

END

pf (continued) **Claim c** Let $f(x) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } x_1 \leq x_2 \\ 0 & \text{o.w.} \end{cases}$ $g(x) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } x_1 \leq x_3 \\ 0 & \text{o.w.} \end{cases}$

Then $f, g : [N]^n \longrightarrow \{0,1\}$ are two increasing functions.

pf of claim c: $x \leq y$ and $f(x)=1 \Rightarrow x \leq y$ and $x_1 \leq x_2 \Rightarrow x_2 - x_1 \leq y_2 - y_1$ and $x_1 \leq x_2$

Therefore we arrive at $y_1 \leq y_2$ and hence $f(y)=1$.

END

Claim d. Let $\mu : [N]^n \longrightarrow \mathbb{R}_{\geq 0}$ s.t. $\mu(x) = \begin{cases} 1 & \text{if } x \text{ satisfies the inequalities in } \Gamma \\ 0 & \text{o.w.} \end{cases}$

Then μ is log-supermodular.

pf of claim d: For $z, y \in [N]^n$, if $\mu(z) = \mu(y) = 1$ and $x_i < x_j$ is one of the inequalities in Γ. Then we have $z_i < z_j$ & $y_i < y_j$.

Hence $(z \wedge y)_i = \min(z_i - z_1, y_i - y_1) + \max(z_1, y_1)$

$$\leq \min(z_j - z_1, y_j - y_1) + \max(z_1, y_1) = (z \wedge y)_j$$

Similarly, $(z \vee y)_i \leq (z \vee y)_j$. Thus $\mu(z \wedge y) = \mu(z \vee y) = 1$.

Therefore $\mu(z)\mu(y) \leq \mu(z \wedge y)\mu(z \vee y)$.

END

pf (continued)

Claims a, b, c, d + FKG inequality

$$\Rightarrow \left(\sum_{x\in[N]^n}\mu(x)f(x)\right)\left(\sum_{x\in[N]^n}\mu(x)g(x)\right)\le\left(\sum_{x\in[N]^n}\mu(x)f(x)g(x)\right)\left(\sum_{x\in[N]^n}\mu(x)\right)$$

$$\Rightarrow P(x_1\le x_2, \Gamma)\, P(x_1\le x_3, \Gamma)\le P(x_1\le x_2,\ x_1\le x_3, \Gamma)\, P(\Gamma)$$

$\Rightarrow$ As $N\to\infty$ we note that $P(x_i=x_j)\to 0$ provided $i\neq j$, and hence

$$P(x_1<x_2, \Gamma)\, P(x_1<x_3, \Gamma)\le P(x_1<x_2,\ x_1<x_3, \Gamma)\, P(\Gamma)$$

$$\Rightarrow P(x_1<x_2 \mid \Gamma)\le P(x_1<x_2 \mid x_1<x_3, \Gamma)$$

QED

Remark: You might note that in the above proof we choose lattice $([N]^n, \le)$ fist, and then let $N\to\infty$ to get what we want. Can you explain why don't we replace $([N]^n, \le)$ by $(S_n, \le)$?

the set of pesmutation on integers $1, 2, \cdots, n$

Linear Extension

- Let $(P, \leq)$ be a poset with n elements.
  A mapping $\sigma : P \xrightarrow{1\text{-}1} [n]$ is called a linear extension of P
  if $x \leq y \Rightarrow \sigma(x) \leq \sigma(y)$
- In the remaining slides, we consider the probability space of all linear extension of P, where each sample point is equally likely.

Example: Consider a Hasses diagram of 4-elements as follows.

$\Rightarrow \exists\, 6$ linear extension $\begin{matrix}2 & 4\\ | & | \\ 1 & 3\end{matrix}$, $\begin{matrix}3 & 4\\ | & | \\ 1 & 2\end{matrix}$, $\begin{matrix}4 & 3\\ | & | \\ 1 & 2\end{matrix}$, $\begin{matrix}3 & 4\\ | & | \\ 2 & 1\end{matrix}$, $\begin{matrix}4 & 3\\ | & | \\ 2 & 1\end{matrix}$, $\begin{matrix}4 & 2\\ | & | \\ 3 & 1\end{matrix}$.

$P(x \leq y) \stackrel{\text{def}}{=\!=}$ the probability that $\sigma(x) \leq \sigma(y)$ in a randomly chosen linear extension σ.

For example, $P(c \leq b) = P(\{\sigma : \sigma(c) \leq \sigma(b)\}) = \frac{5}{6}$, $P(c \leq d) = \frac{6}{6}$.

The XYZ-Theorem (II)

Thm 6.4.1 [p88] Let P be a poset with n elements $a_1, a_2, \cdots, a_n$. Then

$$P(a_1 \le a_2 \text{ and } a_1 \le a_3) \le P(a_1 \le a_2)\, P(a_1 \le a_3)$$

A false conjecture: Is $P(x_1 < x_2 < x_4 \mid \Gamma) \le P(x_1 < x_2 < x_4 \mid \Gamma, x_1 < x_3 < x_4)$?

Sol: In general its not true!

Let $n = 6$ and $\Gamma = \{x_2 < x_5 < x_6 < x_3,\ x_1 < x_4\}$

Then we have $P(x_1 < x_2 < x_4 \mid \Gamma) = \frac{4}{15}$.

However $P(x_1 < x_2 < x_4 \mid \Gamma, x_1 < x_3 < x_4) = \frac{1}{4}$.

END