# The XYZ-Theorem (I)

**Thm** (Shepp 1982) Let the incomes $x_1, \dots, x_n$ of $n$ individuals be initially ordered at random uniformly on all permutations. Suppose some partial information is available on the true ordering of the $x$'s; e.g. $\Gamma = \{x_1 < x_{12},\ x_7 < x_5, \dots\}$. Then for any $\Gamma$, we have

$$P(x_1 < x_2 \mid \Gamma) \le P(x_1 < x_2 \mid \Gamma,\ x_1 < x_3).$$

**pf:** $[N]^n \overset{def}{=\!=} \overbrace{[N]\times\cdots\times[N]}^{n \text{ terms}}$

If $x, y \in [N]^n$ then we write $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$.

**Claim a.** $([N]^n, \leq)$ is a lattice, where $x \leq y$ iff $x_1 \geq y_1$ and $x_i - x_1 \leq y_i - y_1$ for $i=1,2,3,\dots,n$.

pf of claim a: Clearly $([N]^n, \leq)$ is a poset and that

$$(x \wedge y)_i = \min(x_i - x_1, y_i - y_1) + \max(x_1, y_1). \quad i=1,2,\dots,n$$

$$(x \vee y)_i = \max(x_i - x_1, y_i - y_1) + \min(x_1, y_1). \quad i=1,2,\dots,n$$

Thus $([N]^n, \leq)$ is a lattice.

END

Note: we need to check that $(x\wedge y)_i, (x \vee y)_i \in [N]$! **Prove it!**

Exercise

**pf** (continued) **Claim b.** $([N]^n, \le)$ is a distributive lattice.

pf of claim b: $(x \wedge (y \vee z))_i$

$$= \min(x_i - x_1,\ (y \vee z)_i - (y \vee z)_1) + \max(x_1,\ (y \vee z)_1)$$

$$= \min(x_i - x_1,\ \max(y_i - y_1,\ z_i - z_1)) + \max(x_1,\ \min(y_1,\ z_1))$$

$$= \max(\min(x_i - x_1,\ y_i - y_1),\ \min(x_i - x_1,\ z_i - z_1)) + \min(\max(x_1,\ y_1),\ \max(x_1,\ z_1))$$

$$= \max((x \wedge y)_i - (x \wedge y)_1,\ (x \wedge z)_i - (x \wedge z)_1) + \min((x \wedge y)_1,\ (x \wedge z)_1)$$

$$= ((x \wedge y) \vee (x \wedge z))_i$$

END

**pf** (continued) **Claim c** Let $f(x) \overset{def}{=} \begin{cases} 1 & \text{if } x_1 \le x_2 \\ 0 & \text{o.w.} \end{cases}$ $g(x) \overset{def}{=} \begin{cases} 1 & \text{if } x_1 \le x_3 \\ 0 & \text{o.w.} \end{cases}$

then $f, g : [N]^n \to \{0,1\}$ are two increasing functions.

pf of claim c: $x \le y$ and $f(x)=1 \Rightarrow x \le y$ and $x_1 \le x_2 \Rightarrow x_2 - x_1 \le y_2 - y_1$ and $x_1 \le x_2$

Therefore we arrive at $y_1 \le y_2$ and hence $f(y)=1$.

END

**Claim d.** Let $\mu : [N]^n \to \mathbb{R}_{\ge 0}$ s.t. $\mu(x) = \begin{cases} 1 & \text{if } x \text{ satisfies the inequalities in } \Gamma \\ 0 & \text{o.w.} \end{cases}$

Then $\mu$ is log-supermodular.

pf of Claim d: For $z, y \in [N]^n$, if $\mu(z) = \mu(y) = 1$ and $x_i < x_j$ is one of the inequalities in $\Gamma$. Then we have $z_i < z_j$ & $y_i < y_j$.

Hence $(z \wedge y)_i = \min(z_i - z_1, y_i - y_1) + \max(z_1, y_1)$

$$\le \min(z_j - z_1, y_j - y_1) + \max(z_1, y_1) = (z \wedge y)_j$$

Similarly, $(z \vee y)_i \le (z \vee y)_j$. Thus $\mu(z \wedge y) = \mu(z \vee y) = 1$.

Therefore $\mu(z)\mu(y) \le \mu(z \wedge y)\mu(z \vee y)$.

END

**pf** (continued)

Claims a, b, c, d + FKG inequality

$$\Rightarrow \left(\sum_{x\in[N]^n}\mu(x)f(x)\right)\left(\sum_{x\in[N]^n}\mu(x)g(x)\right)\leq\left(\sum_{x\in[N]^n}\mu(x)f(x)g(x)\right)\left(\sum_{x\in[N]^n}\mu(x)\right)$$

$$\Rightarrow P(x_1\leq x_2, \Gamma)\,P(x_1\leq x_3, \Gamma)\leq P(x_1\leq x_2, x_1\leq x_3, \Gamma)\,P(\Gamma)$$

$\Rightarrow$ As $N\to\infty$ we note that $P(x_i=x_j)\to 0$ provided $i\neq j$, and hence

$$P(x_1<x_2, \Gamma)\,P(x_1<x_3, \Gamma)\leq P(x_1<x_2, x_1<x_3, \Gamma)\,P(\Gamma)$$

$$\Rightarrow P(x_1<x_2\mid\Gamma)\leq P(x_1<x_2\mid x_1<x_3, \Gamma)$$

**QED**

Remark: You might note that in the above proof we choose lattice $([N]^n, \leq)$ fist, and then let $N\to\infty$ to get what we want. Can you explain why don't we replace $([N]^n, \leq)$ by $(S_n, \leq)$?

the set of permutation on integers $1, 2, \cdots, n$

# Linear Extension

- Let $(P, \leq)$ be a poset with $n$ elements.

  A mapping $\sigma : P \xrightarrow{1\text{-}1} [n]$ is called a linear extension of $P$ if $x \leq y \Rightarrow \sigma(x) \leq \sigma(y)$

- In the remaining slides, we consider the probability space of all linear extension of $P$, where each sample point is equally likely.

**Example**: Consider a Hasses diagram of 4-elements as follows.


$\Rightarrow \exists\, 6$ linear extension $\begin{array}{cc}2&4\\|&|\\1&3\end{array}$, $\begin{array}{cc}3&4\\|&|\\1&2\end{array}$, $\begin{array}{cc}4&3\\|&|\\1&2\end{array}$, $\begin{array}{cc}3&4\\|&|\\2&1\end{array}$, $\begin{array}{cc}4&3\\|&|\\2&1\end{array}$, $\begin{array}{cc}4&2\\|&|\\3&1\end{array}$.

$P(x \leq y) \stackrel{\text{def}}{=\!=}$ the probability that $\sigma(x) \leq \sigma(y)$ in a randomly chosen linear extension $\sigma$.

for example, $P(c \leq b) = P(\{\sigma : \sigma(c) \leq \sigma(b)\}) = \frac{5}{6}$, $P(c \leq d) = \frac{6}{6}$.

# The XYZ-Theorem (II)

**Thm** 6.4.1 [p88] Let $P$ be a poset with $n$ elements $a_1, a_2, \cdots, a_n$. Then

$$P(a_1 \leq a_2 \text{ and } a_1 \leq a_3) \geq P(a_1 \leq a_2)\, P(a_1 \leq a_3)$$

**A false Conjecture:** Is $P(x_1 < x_2 < x_4 \mid \Gamma) \leq P(x_1 < x_2 < x_4 \mid \Gamma, x_1 < x_3 < x_4)$ ?

Sol: In general its not true!

Let $n=6$ and $\Gamma = \{x_2 < x_5 < x_6 < x_3,\ x_1 < x_4\}$

Then we have $P(x_1 < x_2 < x_4 \mid \Gamma) = \frac{4}{15}$.

However $P(x_1 < x_2 < x_4 \mid \Gamma, x_1 < x_3 < x_4) = \frac{1}{4}$.

END