2.1.14 (a):

Proof. ($\Rightarrow$) Suppose that T is one-to-one. Let S be a linearly independent subset of V. We want to show that $T(S)$ is linearly independent. Suppose that $T(S)$ is linearly dependent. Then there exist $v_1, \cdots, v_n \in S$ and some not all zero scalars $a_1, \cdots, a_n$ such that

$$a_1T(v_1) + \cdots + a_nT(v_n) = 0.$$

Since T is linear,

$$a_1T(v_1) + \cdots + a_nT(v_n) = T(a_1v_1 + \cdots + a_nv_n) = 0.$$

By assumption that T is one-to-one, we also know that $N(T) = \{0\}$. Hence

$$a_1v_1 + \cdots + a_nv_n = 0.$$

But S is linearly independent and $v_1, \cdots, v_n \in S$, we have $a_1 = \cdots = a_n = 0$.
$\rightarrow\leftarrow$
Since S is arbitrary, T carries linearly independent subsets of V onto linerly independent subsets of W.
($\Leftarrow$) Suppose that T carries linearly independent subsets of V onto linearly independent subsets of W. Assume that $T(x) = 0$. If the set $\{x\}$ is linearly independent, then by assumption we conclude that $\{0\}$ is linearly independent, which is a contradiction. Hence the set $\{x\}$ is linearly dependent. This implies that $x = 0$. That is, $N(T) = \{0\}$. Therefore, T is one-to-one. □

2.1.14 (b):

Proof. ($\Rightarrow$) Suppose that S is linearly independent. Then, by part (a), we have that $T(S)$ is linearly independent.
($\Leftarrow$) Suppose that $T(S)$ is linearly independent. Assume that S is linearly dependent. Then there exist $v_1, \cdots, v_n \in S$ and some not all zero scalars $a_1, \cdots, a_n$ such that

$$a_1v_1 + \cdots + a_nv_n = 0.$$

Since T is linear, we have

$$0 = T(0) = T(a_1v_1 + \cdots + a_nv_n) = a_1T(v_1) + \cdots + a_nT(v_n).$$

But $T(S)$ is linearly independent, this implies that $a_1 = \cdots = a_n = 0$.
$\rightarrow\leftarrow$
Hence, S is linearly independent. □

2.1.14 (c):

Proof. Suppose that $\beta = \{v_1, \cdots, v_n\}$ is a basis for V and T is one-to-one and onto. We want to show that $T(\beta)$ is linearly independent and $\mathrm{span}(T(\beta)) = W$. Since T is one-to-one and β is linearly independent, by part (b), $T(\beta)$ is linearly independent.
Next, since β is a basis for V, by Theorem 2.2, $R(T) = \mathrm{span}(T(\beta))$. But since T is onto, we have $R(T) = W$. Then $W = \mathrm{span}(T(\beta))$. Therefore $T(\beta)$ is a basis for W. □