**4.3.21:**

*Proof.* Let $A$ be an $k \times k$ matrix.
We proceed by induction on $k$.
For $k = 1$, the cofactor expansion along the first column gives

$$\det(M) = A_{11} \det(\widetilde{M_{11}}) = M_{11} \det(C) = \det(A) \det(C).$$

So the result holds for $k = 1$.
Suppose that the result holds for $k - 1$.
Now, taking the cofactor expansion along the first column gives

$$\begin{aligned} \det(M) &= A_{11} \det(\widetilde{M_{11}}) - \cdots + (-1)^{k+1} A_{k1} \det(\widetilde{M_{k1}}) \\ &= A_{11} \det \begin{pmatrix} \widetilde{A}_{11} & \widetilde{B}_1 \\ 0 & C \end{pmatrix} - \cdots + (-1)^{k+1} A_{k1} \det \begin{pmatrix} \widetilde{A}_{k1} & \widetilde{B}_k \\ 0 & C \end{pmatrix}, \end{aligned}$$

where $\widetilde{B}_i, i = 1, \cdots, k$, is obtained from $B$ by deleting the $i$th row.
Since $\widetilde{A}_{i1}, i = 1, \cdots, k$, is an $(k - 1) \times (k - 1)$ matrix, induction hypothesis implies that

$$\begin{aligned} \det(M) &= A_{11} \det(\widetilde{A}_{11}) \det(C) - \cdots + (-1)^{k+1} A_{k1} \det(\widetilde{A}_{k1}) \det(C) \\ &= \Big( A_{11} \det(\widetilde{A}_{11}) - \cdots + (-1)^{k+1} A_{k1} \det(\widetilde{A}_{k1}) \Big) \det(C) \\ &= \det(A) \det(C), \end{aligned}$$

where the last equality follows from the cofactor expansion of $A$ along the first column.
Thus the result holds for all $k$. □