3.2.14:

Proof of (a). For any $v \in R(T+U)$, we can write it as $v = (T+U)(w) = T(w)+U(w) \in R(T)+R(U)$ for some $w \in V$. Hence $R(T+U) \subseteq R(T)+R(U)$. □

Proof of (b).

$$\begin{aligned}
\operatorname{rank}(T+U) &= \dim(R(T+U)) \\
&\overset{(a)}{\leq} \dim(R(T)+R(U)) \\
&\overset{1.6.29(a)}{=} \dim(R(T)) + \dim(R(U)) - \dim(R(T)\cap R(U)) \\
&\leq \dim(R(T)) + \dim(R(U)) \\
&= \operatorname{rank}(T) + \operatorname{rank}(U).
\end{aligned}$$

□

Proof of (c).

$$\operatorname{rank}(A+B) \overset{def.}{=} \operatorname{rank}(L_{A+B}) = \operatorname{rank}(L_A+L_B) \overset{(b)}{\leq} \operatorname{rank}(L_A)+\operatorname{rank}(L_B) \overset{def.}{=} \operatorname{rank}(A)+\operatorname{rank}(B).$$

□

3.2.15:

Proof. Suppose that A and B are matrices having n rows and M is an $m \times n$ matrix. Let $C = M(A|B)$ and $D = (MA|MB)$. Assume that A has k columns and B has l columns. For $j = 1, \cdots, k$, we have

$$C_{ij} = \sum_{s=1}^{n} M_{is}A_{sj} = (MA)_{ij} = D_{ij},$$

and, for $j = k+1, \cdots, k+l$, we have

$$C_{ij} = \sum_{s=1}^{n} M_{is}B_{sj} = (MB)_{ij} = D_{ij}.$$

Hence $M(A|B) = (MA|MB)$ for any $m \times n$ matrix M. □

3.2.17:

Proof. Let $B \in M_{3\times 1}(F)$, $C \in M_{1\times 3}(F)$, then $\operatorname{rank}(BC) \overset{\text{Theorem3.7}}{\leq} \operatorname{rank}(B) \leq 1$. Conversely, suppose that A is a 3×3 matrix having rank 1, then by Corollary 1 for Theorem 3.5, there exist invertible 3×3 matrixes B_1 and C_1 such that

$$B_1AC_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

This implies that

$$A = B_1^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} C_1^{-1}.$$

Let B be the 3×1 matrix such that $B = B_1^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and C be the 1×3 matrix such that $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} C_1^{-1}$, then $A = BC$. □

3.2.21:

Proof. Suppose that A is an $m\times n$ matrix with rank m. By definition, $m = \operatorname{rank}(A) = \operatorname{rank}(L_A) = \dim R(L_A)$, where $L_A : F^n \to F^m$ is the left multiplication transformation. Let $\beta = \{e_1, e_2, \cdots, e_m\}$ be the standard ordered basis for F^m. Since $m = \dim R(L_A) = \dim F^m$, L_A is surjective. So, for each $i = 1, \cdots, m$, there exists $v_i \in F^n$ such that $L_A(v_i) = Av_i = e_i$. This implies that $A[v_1, \cdots, v_m] = [e_1, \cdots, e_m] = I_m$. Let B be the matrix with column vectors $v_i, i = 1, \cdots, m$, i.e. $B = [v_1, \cdots, v_m]$. Thus B is an $n\times m$ matrix such that $AB = I_m$. □

3.2.22:

Proof. Let B be an $n\times m$ matrix with rank m, then B^t is an $m\times n$ matrix with rank m. By Exercise 3.2.21, there exists an $n\times m$ matrix C such that $B^tC = I_m$. Let $A = C^t$, then $AB = C^tB = (B^tC)^t = (I_m)^t = I_m$. □