Problem 1: Show that a subset $W$ of a vector space $V$ is a subspace of $V$ if and only if $\text{span}(W) = W$. (4 points)

Proof. ($\Rightarrow$) It is clear that $W \subseteq \text{span}(W)$.
We need to show that if $W$ is a subspace of $V$, then $\text{span}(W) \subseteq W$.
For any $u \in \text{span}(W)$,
$$u = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$
for some $v_1, v_2, \cdots, v_n \in W$ and some scalars $a_1, a_2, \cdots, a_n$.
Since $W$ is a subspace of $V$ and $v_1, v_2, \cdots, v_n \in W$,
$$u = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \in W.$$ 
So, $\text{span}(W) \subseteq W$.
Hence, if $W$ is a subspace of $V$, then $W = W$.

($\Leftarrow$) By Theorem 1.5, we have that $\text{span}(W) = W$ is a subspace of $V$. 

Problem 2: Prove that a set $S$ is linearly dependent if and only if $S = \{0\}$ or there exist distinct vectors $v, u_1, u_2, \cdots, u_n$ in $S$ such that $v$ is a linear combination of $u_1, u_2, \cdots, u_n$. (5 points)

Proof. ($\Rightarrow$) If $S$ is linearly dependent and $S \neq \{0\}$, then there exist distinct vectors $u_0, u_1, \cdots, u_n \in S$ such that
$$a_0 u_0 + a_1 u_1 + \cdots + a_n u_n = 0$$
with at least one of the scalars $a_0, a_1, \cdots, a_n$ is not zero, say $a_0 \neq 0$.
Then we have
$$u_0 = \left( -\frac{a_1}{a_0} \right) u_1 + \left( -\frac{a_2}{a_0} \right) u_2 + \cdots + \left( -\frac{a_n}{a_0} \right) u_n.$$ 
Hence $v = u_0$ is a linear combination of $u_1, u_2, \cdots, u_n$.

($\Leftarrow$) If $S = \{0\}$, then it’s clear that $S$ is linearly dependent.
Assume that there exist distinct vectors $v, u_1, u_2, \cdots, u_n \in S$ such that $v$ is a linear combination of $u_1, u_2, \cdots, u_n$, say
$$v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n,$$
for some scalars $a_1, a_2, \cdots, a_n$.
Then we have
$$0 = (-1)v + a_1 u_1 + a_2 u_2 + \cdots + a_n u_n.$$ 
Hence $S$ is linearly dependent. 

Problem 3:

(1) Give an example in which span($S_1 \cap S_2$) and span($S_1$) $\cap$ span($S_2$) are not equal. (3 points)

Solution of (1). For example, let $S_1 = \{(1, 0)\}$ and $S_2 = \{(2, 0)\}$, then

$$\text{span}(S_1 \cap S_2) = \text{span}(\emptyset) = \{(0, 0)\}$$

and

$$\text{span}(S_1) \cap \text{span}(S_2) = x - \text{axis}.$$

(2) Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the functions defined by $f(t) = e^t$ and $g(t) = e^{2t}$.

Prove that $f$ and $g$ are linearly independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$. (3 points)

Solution of (2). Let

$$ae^t + be^{2t} = 0,$$

where $a, b \in \mathbb{R}$. Differentiate the equation with respect to $t$ on both sides, we obtain

$$ae^t + 2be^{2t} = 0.$$

By solving the system of the equations, we obtain $a = b = 0$. Hence $f$ and $g$ are linearly independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$. 