NAME:____________ ID NO.:______________________ CLASS: __________

Problem 1: Let $A = \begin{pmatrix} 4 & 3 \\ 5 & 6 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$,

(1) (4 points) Determine all the eigenvalues and the corresponding eigenvectors of A.

solution. Eigenvalue : $\lambda_1 = 1$, Eigenvector : $v_1 = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}, t \neq 0, t \in \mathbb{R}$.

Eigenvalue : $\lambda_2 = 9$, Eigenvector : $v_2 = t\begin{pmatrix} 3 \\ 5 \end{pmatrix}, t \neq 0, t \in \mathbb{R}$. □

(2) (2 points) Determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

Solution. $Q = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}$. □

(3) (4 points) Use (1) and (2) to compute e^A.

Proof. $\begin{pmatrix} \frac{5e+3e^9}{8} & \frac{-3e+3e^9}{8} \\ \frac{-5e+5e^9}{8} & \frac{3e+5e^9}{8} \end{pmatrix}$. □

Problem 2:

(1) (2 points) Let $T \in \mathcal{L}(V)$ and $\dim(V) < \infty$. Let W be the T-cyclic subspace of V generated by a vector $v \in V \setminus \{0\}$, and $\dim(W) = 3$. Suppose that $-4I(v) + 3T(v) - 2T^2(v) + T^3(v)$ is a zero vector of V. Find the characteristic polynomial $P_{T_W}(t)$ of T_W.

Solution. $-t^3 + 2t^2 - 3t + 4$. □

(2) (4 points) Let T be the linear operator on $M_{2\times 2}(\mathbb{R})$ such that $T(A) = A^t$. Find the characteristic polynomial $P_T(t)$ of T.

Solution. $(t-1)^3(t+1)$. □

Problem 3: Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear operator defined by $T(f(x)) = f(x) + (1+x)f'(x)$, where $P_2(\mathbb{R})$ is the set of all polynomials with real coefficients with degree at most 2 and $f'(x)$ is the derivative of $f(x)$.

(1) (3 points) Find all the eigenvalues of the operator T.

solution. $1, 2, 3$. □

(2) (2 points) Find all the eigenvalues of the operator $T^5 + 2T^3 + 5T$.

Solution. $1^5 + 2 \cdot 1^3 + 5 \cdot 1, 2^5 + 2 \cdot 2^3 + 5 \cdot 2, 3^5 + 2 \cdot 3^3 + 5 \cdot 3$. □

(3) (3 points) Find a basis β for $P_2(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix.

Solution. $\beta = \{1, 1 + x, 1 + 2x + x^2\}$. □

Problem 4: (5 points) Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear operator defined by $T(f(x)) = f'(x) + f''(x)$. Test T for diagonalizability.

Solution. Not diagonalizable. □

Problem 5: (4 points) Let T be a linear operator on an inner product space V, and suppose that $\|T(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

Proof. Let $x \in V$ be an arbitrary vector. Then $T(x) = 0 \Rightarrow \|T(x)\| = \|x\| = 0 \Rightarrow x = 0$. Hence T is one-to-one. □

Problem 6: (4 points) Let $V = C([0, 1])$, and define

$$< f, g >= \int_0^{3/4} f(t)g(t)dt.$$

Is this an inner product on V.

Solution. Let $f(x) = 0$, if $x \leq 3/4$ and $f(x) = x - 3/4$ if $x > 3/4$. Then $< f, f >= 0$, but $f \neq 0$. Hence it is not an inner product on V. □

Problem 7: (5 points) Prove that similar matrices have the same characteristic polynomial.

Proof. Assume that the $n \times n$ matrix A is similar to the $n \times n$ matrix B, then there exists an invertible $n \times n$ matrix Q such that $B = Q^{-1}AQ$. Now

$$\begin{aligned} \det(B - \lambda I) &= \det(Q^{-1}AQ - \lambda I) \\ &= \det(Q^{-1}(A - \lambda I)Q) \\ &= \det(Q^{-1}) \det(A - \lambda I) \det(Q) \\ &= \det(A - \lambda I). \end{aligned}$$

Hence the similar matrices A and B have the same characteristic polynomial. □

Problem 8: Show that if a matrix A is diagonalizable, then

(1) (4 points) the determinant of A, $\det(A)$, is the product of its eigenvalues (counting with multiplicities).

Proof. Since A is diagonalizable, A is similar to a diagonal matrix whose diagonal entries are eigenvalues. Since similar matrices have the same determinant. The determinant of A, $\det(A)$, is equal to the product of its eigenvalues (counting with multiplicities). □

(2) (4 points) the trace of A, $\operatorname{tr}(A)$, is the sum of its eigenvalues (counting with multiplicities).

Proof. Since A is diagonalizable, A is similar to a diagonal matrix whose diagonal entries are eigenvalues. Since similar matrices have the same trace. The trace of A, $\operatorname{tr}(A)$, is equal to the sum of its eigenvalues (counting with multiplicities). □