Problem 1: Let $V$ be an inner product space, and let $T$ be a linear operator on $V$. Prove the following results.

(1) (5 points) $R(T^*)^\perp = N(T)$.

Proof. If $x \in R(T^*)^\perp$, then $0 = <x, T^*(y)> = <T(x), y>$ for any $y \in V$. This implies that $T(x) = 0$, i.e. $x \in N(T)$. Hence $R(T^*)^\perp \subseteq N(T)$.

If $x \in N(T)$, then $0 = <0, y> = <T(x), y> = <x, T^*(y)>$ for any $y \in V$. This implies that $x \in R(T^*)^\perp$. Hence $N(T) \subseteq R(T^*)^\perp$. Therefore, we conclude that $R(T^*)^\perp = N(T)$. \hfill \Box$

(2) (1 point) If $V$ is finite-dimensional, then $R(T^*) = N(T)^\perp$. (Hint: Use the fact that if $W$ is a subspace of a finite-dimensional inner product space $V$, then $W = (W^\perp)^\perp$.)

Proof. By hint, $N(T)^\perp = (R(T^*)^\perp)^\perp = R(T^*)$. \hfill \Box$

(3) (4 points) If $V$ is finite-dimensional, then $N(T^*T) = N(T)$. Deduce that $\text{rank}(T^*T) = \text{rank}(T)$.

Proof. If $x \in N(T^*T)$, then $T^*T(x) = 0$ and $0 = <T^*T(x), x> = <T(x), T(x)>$. This implies that $T(x) = 0$, i.e. $x \in N(T)$. Hence $N(T^*T) \subseteq N(T)$. Conversely, if $x \in N(T)$, then $T^*T(x) = T^*(0) = 0$. This implies that $x \in N(T^*T)$. Hence $N(T) \subseteq N(T^*T)$. We conclude that $N(T) = N(T^*T)$. Finally, by dimension theorem, we have $\dim V = \dim N(T) + \dim \text{rank}(T) = \dim N(T^*T) + \dim \text{rank}(T^*T)$. Therefore, $\text{rank}(T^*T) = \text{rank}(T)$. \hfill \Box$

(4) (3 points) If $V$ is finite-dimensional, then $\text{rank}(T) = \text{rank}(T^*)$.

Proof. By theorem, we have $\dim V = \dim N(T) + \dim N(T)^\perp$. By dimension theorem, we know that $\dim V = \dim N(T) + \dim R(T)$. Hence $\dim N(T)^\perp = \dim R(T)$. Also, we know that, by (b), $\dim N(T)^\perp = \dim R(T^*)$. Therefore, $\dim R(T) = \dim R(T^*)$, i.e. $\text{rank}(T) = \text{rank}(T^*)$. \hfill \Box$

Problem 2: Give an example of a linear operator $T$ on $\mathbb{R}^2$ and an ordered basis $\beta$ for $\mathbb{R}^2$ such that $T$ is normal, but $[T]_\beta$ is not normal.

(1) (3 points) Write down your example.

Solution. For example, let $T(a, b) = (a, 2b)$ and $\beta = \{(1, 1), (0, 1)\}$. \hfill \Box
(2) (3 points) Show that the $T$ in your example is normal.

Solution. Check that $T$ is self-adjoint. Hence $T$ is normal.

(3) (3 points) Show that the $[T]_\beta$ in your example is not normal.

Solution. $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$.

Problem 3: Let $T \in L(V)$, $V = \mathbb{C}^2$ and $F = \mathbb{C}$ such that

$$T(a_1, a_2) = (3i a_1 + 4 a_2, 2a_1 - a_2).$$

Let $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be an ordered basis for $\mathbb{C}^2$.

(1) (3 points) Compute $([T]_\beta)^*$.

Solution. $\begin{pmatrix} -3i + 8 & 6i - 16 \\ 4 & -9 \end{pmatrix}$.

(2) (4 points) Compute $[T^*]_\beta$.

Solution. $< (a_1, a_2), T^*(1, 2) >= < T(a_1, a_2), (1, 2) >= \cdots = < (a_1, a_2), (-3i + 4, 2) >$. $\Rightarrow T^*(1, 2) = (-3i + 4, 2)$. Similarly, $T^*(0, 1) = (2, -1)$. It is easy to show that $[T^*]_\beta = \begin{pmatrix} -3i + 4 & 2 \\ 6i - 6 & -5 \end{pmatrix}$.

Problem 4: (4 points) Prove that a $3 \times 3$ matrix that is both unitary and upper triangular must be a diagonal matrix.

Proof. It can be proved by straightforward computation. We skip the details.

Problem 5: (5 points) Let $A$ be an $n \times n$ real symmetric or complex normal matrix. Prove that

$$\text{tr}(A^*A) = \sum_{i=1}^n |\lambda_i|^2,$$

where the $\lambda_i$'s are the (not necessarily distinct) eigenvalues of $A$.

Proof. By Theorem 6.19 and Theorem 6.20, $A$ is similar to a diagonal matrix whose diagonal entries consist of eigenvalues, i.e. there exists an invertible matrix $Q$ such that $Q^{-1}AQ = D = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Hence $\text{tr}(A^*A) = \text{tr}((QDQ^{-1})^*(QDQ^{-1})) = \text{tr}(QD^*DQ) = \text{tr}(D^*D) = \sum_{i=1}^n |\lambda_i|^2$.

Problem 6: Let $V = P(\mathbb{R})$ with the inner product $< f(x), g(x) > = \int_{-1}^1 f(t)g(t)dt$, and consider the subspace $P_2(\mathbb{R})$ with the standard ordered basis $\beta = \{1, x, x^2\}$.
(1) (5 points) Use the Gram-Schmidt process to replace $\beta$ by an orthogonal basis $
abla{v_1, v_2, v_3}$ for $P_2(\mathbb{R})$.

Solution. $\{1, x, x^2 - 1/3\}$. □

(2) (3 points) Use the orthogonal basis in (1) to obtain an orthonormal basis for $P_2(\mathbb{R})$.

Solution. $\left\{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}x, \frac{\sqrt{5}}{8}(3x^2 - 1)\right\}$. □

Problem 7: (4 points) Let $V$ be a finite dimensional inner product space, and $X$ and $Y$ be two subspaces of $V$. If $\dim X < \dim Y$, show that there exists a nonzero vector $y$ in $Y$ such that $y$ is orthogonal to all vectors in $X$.

Proof. If $y$ is $Y$ such that $y$ is orthogonal to all vectors in $X$, then $y \in X^\perp \cap Y$. So it is enough to show that $\dim(X^\perp \cap Y) > 0$. We know that $X^\perp + Y$ is a subspace of $V$, so $\dim(X^\perp + Y) \leq \dim V$. We also know that $\dim V = \dim X + \dim X^\perp$ and $\dim(X^\perp + Y) = \dim X^\perp + \dim Y - \dim(X^\perp \cap Y)$. Hence, by assumption and the above equalities, we have

$$\dim(X^\perp \cap Y) = \dim X^\perp + \dim Y - \dim(X^\perp + Y)$$

$$= (\dim V - \dim X) + \dim Y - \dim(X^\perp + Y)$$

$$\geq \dim Y - \dim X > 0.$$ □