NAME:____________ ID NO.:____________________ CLASS: __________

Problem 1:(10 points) Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$.

(1) Find $e^{At}e^{Bt}$ and $e^{(A+B)t}$.

Solution.

$$e^{(A+B)t} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad e^{Bt} = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}.$$

Hence

$$e^{At}e^{Bt} = \begin{pmatrix} 1-t^2 & t \\ -t & 1 \end{pmatrix}.$$

□

(2) Are they equal?

Solution. Obviously $e^{At}e^{Bt} \neq e^{(A+B)t}$. □

Problem 2:(10 points) Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

(1) Prove that if A is invertible, then

$$A^{-1} = (-1/a_0)[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n].$$

Proof. Cayley-Hamilton Theorem says that

$$(-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n = 0.$$

Hence

$$\begin{aligned} &A\left\{(-1/a_0)[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n]\right\} \\ &= (-1/a_0)[(-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A] \\ &= (-1/a_0)(-a_0 I_n) = I_n. \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} &\left\{(-1/a_0)[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n]\right\} A \\ &= (-1/a_0)[(-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A] \\ &= (-a_0 I_n)(-1/a_0) = I_n. \end{aligned}$$

Therefore,

$$A^{-1} = (-1/a_0)[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n].$$

□

(2) Use (1) to compute A^{-1} for

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Solution. Since C.P. of $A = \det(A - tI) = -t^3 + 2t - 1$. By Cayley-Hamilton Theorem, we have

$$-A^3 + 2A - I = 0 \Rightarrow -A^3 + 2A = I \Rightarrow A(-A^2 + 2I) = I.$$

Hence

$$A^{-1} = -A^2 + 2I = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

□