NAME:____________ ID NO.:______________________ CLASS: __________

Problem 1: Let V be an inner product space, and let T be a linear operator on V. Prove the following resluts

(1) (5 points) $R(T^*)^\perp = N(T)$.

Proof. If $x \in R(T^*)^\perp$, then $0 =< x, T^*(y) >=< T(x), y >$ for any $y \in V$. This implies that $T(x) = 0$, i.e. $x \in N(T)$. Hence $R(T^*)^\perp \subseteq N(T)$.

If $x \in N(T)$, then $0 =< 0, y >=< T(x), y >=< x, T^*(y) >$ for any $y \in V$. This implies that $x \in R(T^*)^\perp$. Hence $N(T) \subseteq R(T^*)^\perp$. Therefore, we conclude that $R(T^*)^\perp = N(T)$. □

(2) (1 point) If V is finite-dimensional, then $R(T^*) = N(T)^\perp$. (Hint: Use the fact that if W is a subspace of a finite-dimensional inner product space V, then $W = (W^\perp)^\perp$.)

Proof. By hint, $N(T)^\perp = \left(R(T^*)^\perp\right)^\perp = R(T^*)$. □

(3) (4 points) If V is finite-dimensional, then $\text{rank}(T) = \text{rank}(T^*)$. (Hint: Use the Dimension theorem and (2).)

Proof. By theorem, we have $\dim V = \dim N(T) + \dim N(T)^\perp$. By dimension theorem, we know that $\dim V = \dim N(T) + \dim R(T)$. Hence $\dim N(T)^\perp = \dim R(T)$. Also, we know that, by (b), $\dim N(T)^\perp = \dim R(T^*)$. Therefore, $\dim R(T) = \dim R(T^*)$, i.e. $\text{rank}(T) = \text{rank}(T^*)$. □

Problem 2:(5 points) For the inner product space $V = \mathbb{C}^2$ and the linear operator $T(z_1, z_2) = (2z_1 + iz_2, (1-i)z_1)$ on V, evaluate T^* at the given vector $x = (3 - i, 1 + 2i)$ in V.

Solution. $T^*(x) = (5 + i, -1 - 3i)$. □

Problem 3:(5 points) Find the minimal solution to the following system of linear equations.

$$\begin{aligned} x + y + z - w &= 1 \\ 2x - y \quad\quad + w &= 1 \end{aligned}$$

Solution. This is Exercise 6.3.22.(d). For solution, see page 582 in the textbook. □