NAME:____________ ID NO.:____________________ CLASS: __________

Problem 1: Give an example of a linear operator T on $\mathbb{R}^2$ and an ordered basis β for $\mathbb{R}^2$ such that $[T]_\beta$ is normal, but T is not normal.

(1) (4 points) Write down your example.

Example: Let $\beta = \{(1,1),(0,1)\}$ be an ordered basis for $\mathbb{R}^2$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(a,b) = (0, a-b)$. □

(2) (3 points) Show that $[T]_\beta$ is normal in your example.

Solution. $[T]_\beta = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$. Since $[T]_\beta$ ia a diagonal matrix, hence $[T]_\beta$ is normal. □

(3) (3 points) Show that T is not normal in your example.

Solution. $T^*(c,d) = (d,-d)$.
$TT^*(a,b) = (0,2b) \neq (a-b,-a+b) = T^*T(a,b)$. □

Problem 2: (5 points) Let T and U be self-adjoint operators on an inner product space V. Prove that TU is self-adjoint if and only if $TU = UT$.

Proof. $\Rightarrow$ Suppose that TU is self-adjoint. Since T and U are self-adjoint, $TU = (TU)^* = U^*T^* = UT$.
$\Leftarrow$ Suppose that $TU = UT$. Since T and U are self-adjoint, $TU = T^*U^* = (UT)^* = (TU)^* = U^*T^* = UT$. □

Problem 3: (5 points) Suppose that $V = \mathbb{C}^2$ and T is defined by $T(a,b) = (2a+ib, a+2b)$. Determine whether T is normal, self-adjoint, or neither.

Solution. T is normal, but not self-adjoint. □