Outline of the short course

1. Mathematical background
   - Total variation and $BV(\Omega)$ space
   - Calculus of variations and Euler-Lagrange equation

2. Variational image denoising
   - Rudin-Osher-Fatemi (ROF) model
   - Nonlinear PDE-based denoising algorithm

3. Numerical method: split Bregman iterations

4. Variational image contrast enhancement

5. Final project on variational image stitching
   - Image alignment
   - Image blending
Let $\Omega = (a, b) \subset \mathbb{R}$ be an open (bounded) interval. Let $\mathcal{P}_n = \{x_0, x_1, \cdots, x_{n-1}, x_n\}$, with $x_0 = a$ and $x_n = b$, be an arbitrary partition of $[a, b]$ and $\Delta x_i = x_i - x_{i-1}$, for $i = 1, 2, \cdots, n$. The total variation of a real-valued function $u : \overline{\Omega} \rightarrow \mathbb{R}$ is defined as the quantity,

$$
\|u\|_{TV(\Omega)} := \sup_{\mathcal{P}_n} \sum_{i=1}^{n} |u(x_i) - u(x_{i-1})| \\
= \sup_{\mathcal{P}_n} \sum_{i=1}^{n} \left| \frac{u(x_i) - u(x_{i-1})}{\Delta x_i} \right| \Delta x_i \\
= \int_{\Omega} |u'(x)| \, dx, \text{ provided } u \text{ is a smooth function.}
$$

$\|u\|_{TV(\Omega)}$ is not a norm on any vector space because $\|u\|_{TV(\Omega)} = 0$ does not imply $u \equiv 0$. In fact, any constant function $u$ has $\|u\|_{TV(\Omega)} = 0$.

If $\|u\|_{TV(\Omega)} < \infty$, then we say that $u$ is a function of bounded variation.
Examples of bounded variation functions

All these three functions $f$, $g$ and $h$ have total variation 2
Denoising

Total variation of $u = \|u\|_{TV(\Omega)} = \int_{\Omega} |u'(x)| \, dx$ if $u$ is smooth.

minimizes $\left( \int_{\Omega} |u'(x)| \, dx + \text{some data fidelity term} \right) \implies \text{denoising!}$
The bounded variation space $BV(\Omega)$

Let $\Omega$ be an open subset of $\mathbb{R}^2$. The space of functions of bounded variation $BV(\Omega)$ is defined as the space of real-valued function $u \in L^1(\Omega)$ such that the total variation is finite, i.e.,

$$BV(\Omega) = \{ u \in L^1(\Omega) : \|u\|_{TV(\Omega)} < \infty \},$$

where

$$\|u\|_{TV(\Omega)} := \sup \left\{ \int_{\Omega} u(\nabla \cdot \varphi) \, dx : \varphi \in C_c^1(\Omega, \mathbb{R}^2), \|\varphi\|_{(L^\infty(\Omega))^2} \leq 1 \right\},$$

$C_c^1(\Omega, \mathbb{R}^2)$ is the space of continuously differentiable vector functions with compact support in $\Omega$, $L^1(\Omega)$ and $L^\infty(\Omega)$ are the usual $L^p(\Omega)$ space for $p = 1$ and $p = \infty$, respectively.

*Then $BV(\Omega)$ is a Banach space with the norm,

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \|u\|_{TV(\Omega)}.$$*
The ROF total-variation regularization model

Let \( f : \overline{\Omega} \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be a given noisy image. Rudin, Osher, and Fatemi (\textit{Physica D}, 1992) proposed the model for image denoising:

\[
\min_{u \in BV(\Omega) \cap L^2(\Omega)} \left( \|u\|_{TV(\Omega)} + \frac{\lambda}{2} \int_{\Omega} (u(x) - f(x))^2 \, dx \right),
\]

where \( \lambda > 0 \) is a tuning parameter which controls the regularization strength. Notice that

- A smaller value of \( \lambda \) will lead to a more regular solution.
- The space of functions with bounded variation help remove spurious oscillations (noise) and preserve sharp signals (edges).
- The \( TV \) term allows the solution to have discontinuities.
The existence, uniqueness and stability of solution

**Theorem:** If $u$ is smooth, then $\|u\|_{TV(\Omega)} = \int_{\Omega} |\nabla u| \, dx$.

**Proof.** For a smooth function $u \in C^1(\Omega)$, or in fact a function $u \in W^{1,1}(\Omega)$, we have

$$- \int_{\Omega} u(\nabla \cdot \varphi) \, dx = \int_{\Omega} \varphi \cdot \nabla u \, dx.$$ 

The sup over all $\varphi \in C^1_c(\Omega, \mathbb{R}^2)$ with $\|\varphi\|_{(L^\infty(\Omega))^2} \leq 1$ is $\int_{\Omega} |\nabla u| \, dx$.

**Theorem:** If $f \in L^2(\Omega)$, the minimizer exists and is unique and is stable in $L^2$ with respect to perturbations in $f$.

**ROF model for image denoising:** Below we assume that $u$ is smooth,

$$\min_u \left( \int_{\Omega} |\nabla u| \, dx + \frac{\lambda}{2} \int_{\Omega} (u(x) - f(x))^2 \, dx \right).$$

$$:= E[u]$$
Calculus of variations

- Calculus of variations is a field of mathematical analysis that deals with maximizing or minimizing functionals, which are mappings from a set of functions to the real numbers.
- Functionals are often expressed as definite integrals involving functions and their derivatives.
- The interest is in extremal functions that make the functional attains a maximum or minimum value.
- The extrema of functionals may be obtained by finding functions where the functional derivative is equal to zero. This leads to solving the associated Euler-Lagrange equation.

– excerpted from Wikipedia
Calculus of variations: a necessary condition

Let \([a, b] \subset \mathbb{R}\) be a given interval. We consider the functional,

\[ E[v] := \int_a^b L(x, v(x), v'(x)) \, dx, \]

where we assume that \(v \in C^2[a, b]\) and \(L \in C^2\) with respect to its arguments \(x, v\) and \(v'\).

- If \(E[v]\) attains a local minimum at \(u\) and \(\eta(x)\) a smooth function vanishes at \(a\) and \(b\), then for \(\epsilon\) close to 0, we have
  \[ E[u] \leq E[u + \epsilon \eta], \]
  where \(\delta u := \epsilon \eta\) is called the variation of \(u\).

- Define \(\Phi(\epsilon) := E[u + \epsilon \eta]\) in the variable \(\epsilon\). Then we have
  \[ \Phi'(0) = \left. \frac{d\Phi}{d\epsilon} \right|_{\epsilon=0} = \int_a^b \left. \frac{dL}{d\epsilon} \right|_{\epsilon=0} \, dx = 0. \] (a necessary condition)
Taking the total derivative of $L(x,v,v')$, where $v = u + \epsilon \eta$ and $v' = u' + \epsilon \eta'$, we have

$$\frac{dL}{d\epsilon} = \frac{\partial L}{\partial v} \frac{dv}{d\epsilon} + \frac{\partial L}{\partial v'} \frac{dv'}{d\epsilon} = \frac{\partial L}{\partial v} \eta + \frac{\partial L}{\partial v'} \eta'.$$

Therefore, we obtain

$$\int_{a}^{b} \frac{dL}{d\epsilon} \bigg|_{\epsilon=0} \ dx = \int_{a}^{b} \left( \frac{\partial L}{\partial u} \eta + \frac{\partial L}{\partial u'} \eta' \right) \ dx = \int_{a}^{b} \left( \frac{\partial L}{\partial u} \eta - \eta \frac{d}{dx} \frac{\partial L}{\partial u'} \right) \ dx,$$

where $L(x,v,v') \rightarrow L(x,u,u')$ when $\epsilon = 0$ and we have employed the integration by parts and the fact that $\eta(a) = \eta(b) = 0$. Then, we finally have

$$\int_{a}^{b} \eta \left( \frac{\partial L}{\partial u} - \frac{d}{dx} \frac{\partial L}{\partial u'} \right) \ dx = 0$$

for all smooth functions $\eta$'s vanish at $a$ and $b$. 

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Variational Methods for Image Processing – 11/40
The Euler-Lagrange equation

Fundamental lemma of calculus of variations: If $F$ is a continuous function on an open interval $(a, b)$ and satisfies

$$\int_{a}^{b} F(x)G(x) \, dx = 0$$

for all compactly supported smooth functions $G$ on $(a, b)$, then $F$ is identically zero on $(a, b)$. If $(a, b)$ is replaced by the closed interval $[a, b]$, then we require only that $G$ vanishes at the endpoints $a$ and $b$.

According to the above fundamental lemma, we have

$$\frac{\delta E}{\delta u} := \frac{\partial L}{\partial u} - \frac{d}{dx} \frac{\partial L}{\partial u'} = 0 \quad \text{in} \quad (a, b),$$

which is called the Euler-Lagrange equation, and where $\frac{\delta E}{\delta u}$ is called the functional derivative of $E[u]$.
The Euler-Lagrange equations of some models

Consider the regularized minimization problem:

$$\min_{u \in V} \left( F(u) + \frac{\lambda}{2} \int_{\Omega} (u(x) - f(x))^2 \, dx \right),$$

where $V$ is a suitable space and $\lambda > 0$ is the regularization parameter. Below are the associated E-L equations of three different regularizers:

- **ROF regularizer:** $F(u) = \int_{\Omega} |\nabla u| \, dx = \int_{\Omega} \sqrt{u_x^2 + u_y^2} \, dx$

  $$-\nabla \cdot \frac{\nabla u}{|\nabla u|} + \lambda u = \lambda f \quad \text{in } \Omega. \quad \text{(nonlinear)}$$

- **Tikhonov quadratic regularizer:** $F(u) = \frac{1}{2} \int_{\Omega} u^2 \, dx$

  $$u + \lambda u = \lambda f \quad \text{in } \Omega \implies u = \frac{\lambda}{1+\lambda} f \quad \text{in } \Omega. \quad \text{(linear)}$$

- **Tikhonov quadratic regularizer:** $F(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx$

  $$-\nabla \cdot \nabla u + \lambda u = \lambda f \quad \text{in } \Omega \implies -\frac{1}{\lambda} \Delta u + u = f \quad \text{in } \Omega. \quad \text{(linear)}$$
Numerical results of the Tikhonov quadratic models

From left to right: $u, f, F(u) = \frac{1}{2} \int_{\Omega} u^2 \, dx$, and $F(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx$

(all images credit to Pei-Chiang Shao)

- In the first model, there is no regularization of any kind, since $u = (\lambda f) / (1 + \lambda)$ in $\Omega$. Obviously, this is a wrong choice.

- In the second model, the function space is $V := H^1(\Omega)$. However, there is too much regularization. In fact, the image $u$ belongs to $H^1(\Omega)$, which cannot present discontinuities such as edges or boundaries of objects.
Nonlinear PDE-based denoising algorithm

The Euler-Lagrange equation of ROF model is given by

$$-\nabla \cdot \frac{\nabla u}{|\nabla u|} + \lambda u = \lambda f \quad \text{in } \Omega.$$  

Since the ROF problem is convex, the steady-state solution of the gradient descent is the minimizer of the problem. Therefore, the minimizer can be obtained numerically by evolving a finite difference approximation of the parabolic partial differential equation with the Neumann boundary condition:

$$\begin{cases}
\frac{\partial u}{\partial t} - \nabla \cdot \frac{\nabla u}{|\nabla u|} + \lambda u = \lambda f & \text{for } (t, x) \in (0, T) \times \Omega, \\
\nabla u \cdot n = 0 & \text{for } t \in [0, T] \text{ and } x \in \partial \Omega \oplus u(0, x) = f(x) \text{ for } x \in \Omega.
\end{cases}$$

\textit{Heat–type equation} 

\textit{Neumann boundary condition} 

\textit{initial condition}
The forward Euler finite difference scheme: $\Omega = (0, 1) \times (0, 1)$

Let $x_i = ih$ and $y_j = jh$, $i, j = 0, 1, \cdots, N$, with $h = 1/N$ and $t_n = n\Delta t$, $n \geq 0$. Let $u_{i,j}^n$ be the difference approximation to $u(x_i, y_j, t_n)$.

\[
\begin{align*}
    u_{i,j}^{n+1} &= u_{i,j}^n + \Delta t \lambda (f_{i,j} - u_{i,j}^n) + \frac{\Delta t}{h} \left\{ \nabla_x^- \left( \frac{\nabla_x^+ u_{i,j}^n}{\sqrt{(\nabla_x^+ u_{i,j}^n)^2 + (m(\nabla_y^+ u_{i,j}^n, \nabla_y^- u_{i,j}^n))^2}} \right) \\
    &\quad + \nabla_y^- \left( \frac{\nabla_y^+ u_{i,j}^n}{\sqrt{(\nabla_y^+ u_{i,j}^n)^2 + (m(\nabla_x^+ u_{i,j}^n, \nabla_x^- u_{i,j}^n))^2}} \right) \right\}, \\
    &\quad 1 \leq i, j \leq N - 1,
\end{align*}
\]

$u_{0,j}^n = u_{1,j}^n$, $u_{N,j}^n = u_{N-1,j}^n$, $u_{i,0}^n = u_{i,1}^n$, $u_{i,N}^n = u_{i,N-1}^n$, $0 \leq i, j \leq N$.

where $m(a, b) := (\frac{\text{sign}a + \text{sign}b}{2}) \min\{|a|, |b|\}$, see [3] for more details.

- The forward Euler scheme is conditionally stable, we need $\Delta t / h^2 \leq c$.
- Numerous algorithms have been proposed to solve the TV denoising minimization problem, e.g., the split Bregman iteration.
Three indices to measure the quality

Below are three indices to measure the quality of images and to evaluate the denoising performance. Let $\tilde{u}$ be the clean image, $\bar{u}$ be the mean intensity of the clean image, and $u$ be the produced image.

$$MSE(\tilde{u}, u) := \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} (\tilde{u}_{i,j} - u_{i,j})^2$$

(mean squared error)

$$PSNR := 10 \log_{10} \left( \frac{255^2}{MSE(\tilde{u}, u)} \right)$$

(peak signal to noise ratio)

$$SNR := 10 \log_{10} \left( \frac{MSE(\tilde{u}, \bar{u})}{MSE(\tilde{u}, u)} \right)$$

(signal to noise ratio)
Numerical results

λ = 30 has the best PSNR (all images credit to Pei-Chiang Shao)

A smaller value of λ implies stronger denoising. When λ is very small, the image becomes cartoon-like with sharp jumps between nearly flat regions.
Numerical experiments

Original image size = 256x256

Noisy image (SNR 1.9306)

TV $\lambda = 1$ (SNR 6.1226)

$H^1 : -\mu \Delta u + u - f = 0$ versus TV: ROF model by split Bregman

(all images credit to Pei-Chiang Shao)
Discretization of the ROF model

Total variation is approximated by \( \| u \|_{TV(\Omega)} \approx h^2 \sum_{i=1}^{N} \sum_{j=1}^{N} |\nabla u_{i,j}| \), where the discrete gradient operator as \( \nabla u_{i,j} = (\nabla^+ x u_{i,j}, \nabla^+ y u_{i,j})^\top \),

\[
\nabla^+ x u_{i,j} = \begin{cases} 
\frac{u_{i,j+1} - u_{i,j}}{h}, & 1 \leq j \leq N - 1, \\
0, & j = N;
\end{cases} \\
\nabla^+ y u_{i,j} = \begin{cases} 
\frac{u_{i+1,j} - u_{i,j}}{h}, & 1 \leq i \leq N - 1, \\
0, & i = N.
\end{cases}
\]

Applying the operator splitting technique, we obtain the constrained approximate minimization of the ROF model:

\[
\min_{d, u} \left( \sum_{i,j} |d_{i,j}| + \frac{\lambda}{2} \sum_{i,j} (f_{i,j} - u_{i,j})^2 \right) \quad \text{subject to } d_{i,j} = \nabla u_{i,j}.
\]

Introducing a penalty parameter \( \gamma > 0 \), we obtain the unconstrained minimization problem:

\[
\min_{d, u} \left( \sum_{i,j} |d_{i,j}| + \frac{\lambda}{2} \sum_{i,j} (f_{i,j} - u_{i,j})^2 + \frac{\gamma}{2} \sum_{i,j} |d_{i,j} - \nabla u_{i,j} - b_{i,j}|^2 \right),
\]

where \( b \) is an auxiliary variable related to the Bregman iteration.
Goldstein and Osher (2009) proposed to solve the above-mentioned problem by an alternating direction approach:

**u-subproblem:** With $d$ fixed, we solve

$$u^{k+1} = \arg\min_u \left( \frac{\lambda}{2} \sum_{i,j} (f_{i,j} - u_{i,j})^2 + \frac{\gamma}{2} \sum_{i,j} |d_{i,j}^k - \nabla u_{i,j} - b_{i,j}^k|^2 \right).$$

The optimal $u$ satisfies a discrete screened Poisson equation,

$$\lambda(u - f) + \gamma \nabla \cdot (\nabla u - d + b) = 0,$$

or equivalently,

$$\lambda u - \gamma \Delta u = \lambda f - \gamma \nabla \cdot (d - b),$$

$\nabla \cdot$ and $\Delta$ are the discrete divergence and Laplacian, respectively.

It can be viewed as the EL equation of the minimization problem:

$$\min_u \frac{\lambda}{2} \int_\Omega (f - u)^2 \, dx + \frac{\gamma}{2} \int_\Omega |d - \nabla u - b|^2 \, dx.$$
The discrete screened Poisson equation

\[ \lambda u - \gamma \Delta u = \lambda f - \gamma \nabla \cdot (d - b), \]

which is a symmetric and diagonally dominant linear system, may be solved for \( u \) in the Fourier domain or by the iterative matrix techniques such as the Gauss-Seidel iterative method:

\[ (\lambda + 4\gamma)u_{i,j}^{k+1} = c_{i,j}^k + \gamma \left( u_{i-1,j}^{k+1} + u_{i+1,j}^{k+1} + u_{i,j-1}^{k+1} + u_{i,j+1}^{k+1} \right), \]

where

\[ c_{i,j}^k := (\lambda f - \gamma \nabla \cdot (d - b))_{i,j}^k. \]
**d-subproblem**

**d-subproblem:** With \( u \) fixed, we solve

\[
d^{k+1} = \arg \min_d \left( \sum_{i,j} |d_{i,j}| + \frac{\gamma}{2} \sum_{i,j} |d_{i,j} - \nabla u_{i,j}^{k+1} - b_{i,j}^k|^2 \right),
\]

which has a closed-form solution,

\[
d_{i,j}^{k+1} = \frac{\nabla u_{i,j}^{k+1} + b_{i,j}^k}{|\nabla u_{i,j}^{k+1} + b_{i,j}^k|} \max \left\{ |\nabla u_{i,j}^{k+1} + b_{i,j}^k| - \frac{1}{\gamma}, 0 \right\}.
\]

**Note:** Consider the simple 1-D case,

\[
\arg \min_x \left( \tau |x| + \frac{\rho}{2} (x - y)^2 \right) = \begin{cases} 
  y - \tau / \rho, & y > \tau / \rho \\
  0, & |y| \leq \tau / \rho \\
  y + \tau / \rho, & y < -\tau / \rho 
\end{cases} \\
= \frac{y}{|y|} \max \left\{ |y| - \tau / \rho, 0 \right\}.
\]
Updating $b$ and selecting $\gamma$

- **Updating $b$:** The auxiliary variable $b$ is initialized to zero and updated as $b^{k+1} = b^k + \nabla u^{k+1} - d^{k+1}$.

- **Selecting $\gamma$:** A good choice of $\gamma$ is one for which both $d$ and $u$ subproblems converge quickly and are numerically well-conditioned.
  - In $d$ subproblem, the shrinking effect is more dramatic when $\gamma$ is small.
  - In $u$ subproblem, the effect of $\Delta$ and $\nabla \cdot$ increase when $\gamma$ gets larger. It is also ill-conditioned in the limit $\gamma \to \infty$.

*Therefore, $\gamma$ should be neither extremely large nor small for good convergence.*
The split Bregman algorithm

The split Bregman algorithm:

\[ \text{initialize } u = f, d = b = 0 \]

\[ \text{while } \|u_{\text{current}} - u_{\text{previous}}\|_2 > \text{tolerance} \text{ do} \]

\[ \text{solve the } u\text{-subproblem} \]

\[ \text{solve the } d\text{-subproblem} \]

\[ b = b + \nabla u - d \]

The default parameter values are: \( \text{tolerance} = \|f\|_2/1000 \) and \( \gamma = 5 \).

Color images (RGB channels): The vectorial TV (VTV) is used in place of TV,

\[ \|u\|_{\text{VTV}(\Omega)} := \int_{\Omega} \left( \sum_{i \in \text{channels}} |\nabla u_i(x)|^2 \right)^{1/2} dx. \]

The grayscale algorithm can be extended directly to VTV-regularized image denoising.


Contrast enhancement

The main purpose of contrast enhancement is to adjust the image intensity to enhance the quality and features of the image for a better human visual perception or machine vision identification.

A low-light image and its enhanced result
Histogram equalization (HE)

- We are given a grayscale image \( f : \Omega \to [0, 1] \). The cumulative histogram \( F \) is defined by considering \( f \) as a random variable,

\[
F(\eta) := P(f \leq \eta) = \frac{1}{|\Omega|} \left| \{ x \in \Omega : f(x) \leq \eta \} \right|, \quad \forall \eta \in [0, 1].
\]

- The histogram equalized image \( u : \Omega \to [0, 1] \) is obtained by defining \( u(x) := F(f(x)) \), which is uniformly distributed provided \( F \) is invertible,

\[
P(F(f) \leq \eta) = P(f \leq F^{-1}(\eta)) = F(F^{-1}(\eta)) = \eta.
\]

Histogram equalization: (left) before; (right) after; (red) corresponding histogram; (black) cumulative histogram (quoted from Wikipedia)
Automatic color equalization (ACE)

We are given a grayscale image \( f : \overline{\Omega} \to [0, 1] \). First, the following operation is performed

\[ \tilde{f}(x) = \sum_{y \in \Omega \setminus \{x\}} s_\alpha \left( \frac{f(x) - f(y)}{\|x - y\|} \right), \quad \forall x \in \Omega. \]

Then \( \tilde{f} \) is rescaled to \([0, 1]\) as

\[ u(x) = \frac{\tilde{f}(x) - \min \tilde{f}}{\max \tilde{f} - \min \tilde{f}}. \]

The slope function \( s_\alpha(t) \):

\[ s_\alpha(t) \]

\[ -1 \quad -\frac{1}{\alpha} \quad \frac{1}{\alpha} \quad 1 \]

\[ -1 \quad 1 \]

\[ t \]

\[ s_\alpha(t) := \min\{\max\{\alpha t, -1\}, 1\} \quad (\alpha > 1). \]
A simple variational model

Let $f : \Omega \to \mathbb{R}$ be a given grayscale image. The Morel-Petro-Sbert model (IPOL 2014) is given by

$$
\min_u \frac{1}{2} \left( \int_\Omega |\nabla u - \nabla f|^2 \, dx + \frac{\lambda}{2} \int_\Omega (u - \bar{u})^2 \, dx \right).
$$

- The constant $\bar{u} := \frac{1}{|\Omega|} \int_\Omega u \, dx$ is the mean value of $u$ over $\Omega$.

  The parameter $\lambda > 0$ balances between detail preservation and variance reduction.

- The data fidelity term preserves image details presented in $f$ and the regularizer reduces the variance of $u$ to eliminate the effect of nonuniform illumination.
Two modified variational models

- The original model is simple but difficult to solve due to the $\bar{u}$ term. Therefore, by assuming that $\bar{u} \approx \bar{f}$, it was simplified to

$$\min_u \frac{1}{2} \int_\Omega |\nabla u - \nabla f|^2 \, dx + \frac{\lambda}{2} \int_\Omega (u - \bar{f})^2 \, dx.$$ 

- Petro-Sbert-Morel (MAA 2014) further improved their model by using the $L^1$ norm to obtain sharper edges:

$$\min_u \int_\Omega |\nabla u - \nabla f| \, dx + \frac{\lambda}{2} \int_\Omega (u - \bar{f})^2 \, dx.$$ 

Note that requiring the desired image $u$ being close to a pixel-independent constant $\bar{f}$ highly contradicts the requirement of $\nabla u$ being close to $\nabla f$ and restrains the parameter $\lambda$ to be very small.
Hsieh-Shao-Yang \textit{(SIIMS 2020)} proposed two adaptive functions \(g\) and \(h\) to replace \(f\) and the original input image \(f\),

\[
\min_u \int_\Omega |\nabla u - \nabla h| \, dx + \frac{\lambda}{2} \int_\Omega (u - g)^2 \, dx + \chi_{[0,255]}(u),
\]

where \(g\) and \(h\) are devised respectively as

\[
g(x) = \begin{cases} 
\alpha f, & x \in \Omega_d, \\
 f(x), & x \in \Omega_b,
\end{cases}
\]

\[
h(x) = \begin{cases} 
\beta f(x), & x \in \Omega_d, \\
 f(x), & x \in \Omega_b,
\end{cases}
\]

\[
\Omega_d := \{ x \in \overline{\Omega} : f(x) \leq \bar{f} \}, \quad \Omega_b := \{ x \in \overline{\Omega} : f(x) > \bar{f} \},
\]

with a brightness parameter \(\alpha > 0\) and a contrast-level parameter \(\beta > 1\), and the characteristic function is defined as

\[
\chi_{[0,255]}(u) = \begin{cases} 
0, & \text{range}(u) \subseteq [0,255], \\
\infty, & \text{otherwise}.
\end{cases}
\]

Generally speaking, \(\Omega_d\) contains relatively dim elements, while \(\Omega_b\) contains relatively bright elements.
The domain division for color RGB images denoted by \( (f_R, f_G, f_B) \) is conducted as follows. First, we define the maximum image as

\[
f_{\text{max}}(x) := \max\{f_R(x), f_G(x), f_B(x)\}, \quad \forall x \in \Omega.
\]

Let \( \bar{f}_{\text{max}} := \frac{1}{|\Omega|} \int_{\Omega} f_{\text{max}} \, dx \). Then we divide the image domain \( \Omega \) into two parts

\[
\Omega_d := \{x \in \overline{\Omega} : f_{\text{max}}(x) \leq \bar{f}_{\text{max}}\},
\]

\[
\Omega_b := \{x \in \overline{\Omega} : f_{\text{max}}(x) > \bar{f}_{\text{max}}\}.
\]

As an example, consider an element \( x^* \in \overline{\Omega} \) with color intensities \( (f_R(x^*), f_G(x^*), f_B(x^*)) = (25, 25, 200) \), then \( f_{\text{max}}(x^*) = 200 \), a large value which should be classified into \( \Omega_b \).
Domain division for color images

(top row): low-light images  (bottom row): domain-division results
Adaptive variational model for color images

- With the help of the maximum image $f_{\text{max}}$, we can now process color images channelwise. For every $f \in \{f_R, f_G, f_B\}$, we solve

$$\min_u \int_{\Omega} |\nabla u - \nabla h_c| \, dx + \frac{\lambda}{2} \int_{\Omega} (u - g_c)^2 \, dx + \chi_{[0,255]}(u),$$

where the adaptive functions $g_c$ and $h_c$ are defined as

$$g_c(x) := \begin{cases} \alpha f, & x \in \Omega_d, \\ f(x), & x \in \Omega_b, \end{cases}$$

and

$$h_c(x) := \begin{cases} \beta f(x), & x \in \Omega_d, \\ f(x), & x \in \Omega_b. \end{cases}$$

- There is no evidence shown that choosing different $\lambda, \alpha$ and $\beta$ for each channel separately can have specific benefit. Therefore, for simplicity, we fix $\lambda, \alpha$, and $\beta$ across channel.
The alternating minimization algorithm

The discrete gradient of \( u \) is defined as \( \nabla u_{i,j} = (\nabla^+_x u_{i,j}, \nabla^+_y u_{i,j}) \),

\[
\nabla^+_x u_{i,j} := \begin{cases} (u_{i,j+1} - u_{i,j})/h, & 1 \leq j \leq N - 1, \\ 0, & j = N, \end{cases}
\]
\[
\nabla^+_y u_{i,j} := \begin{cases} (u_{i+1,j} - u_{i,j})/h, & 1 \leq i \leq N - 1, \\ 0, & i = N, \end{cases}
\]

The continuous model can be discretized as

\[
\min_u \sum_{i,j} |\nabla u_{i,j} - \nabla h_{i,j}| + \frac{\lambda}{2} (u_{i,j} - g_{i,j})^2 + \chi_{[0,255]}(u).
\]

Applying the operator splitting, it is then equivalent to

\[
\min_{u,d,v} \sum_{i,j} \left( |d_{i,j}| + \frac{\lambda}{2} (u_{i,j} - g_{i,j})^2 \right) + \chi_{[0,255]}(v),
\]

subject to \( d = \nabla u - \nabla h \) and \( v = u \).
The split Bregman iterations

- The splitted problem can be solved by using the Bregman iteration (or equivalently the augmented Lagrangian method). Introducing the penalty parameter $\gamma > 0$ and $\delta > 0$, we arrive at the following unconstrained minimization problem:

$$\min_{u,d,v} \sum_{i,j} \left( |d_{i,j}| + \frac{\lambda}{2} (u_{i,j} - g_{i,j})^2 + \frac{\gamma}{2} |d_{i,j} - \nabla u_{i,j} + \nabla h_{i,j} - b_{i,j}|^2 ight.$$

$$+ \frac{\delta}{2} (v_{i,j} - u_{i,j} - c_{i,j})^2 \right) + \chi_{[0,255]}(v),$$

where $b$ and $c$ are the variables related to the Bregman iteration (or equivalently the Lagrange multipliers).

- Then the problem is solved by alternating the search directions of $u$, $d$, and $v$. 

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Variational Methods for Image Processing – 37/40
Numerical results

(top row): low-light images  (bottom row): enhanced images
References


Final project on variational image stitching

1. **Image alignment:**

2. **Image blending:**

References:

- [https://www.csie.ntu.edu.tw/~b97074/vfx_html/hw2.html](https://www.csie.ntu.edu.tw/~b97074/vfx_html/hw2.html)
- [https://sites.google.com/a/umich.edu/eecs442-winter2015/homework/image-stitching](https://sites.google.com/a/umich.edu/eecs442-winter2015/homework/image-stitching)
- [https://github.com/xuwenzhe/EECS442-Image-Stitching](https://github.com/xuwenzhe/EECS442-Image-Stitching)