On direct-forcing immersed boundary projection methods for fluid-solid interaction problems

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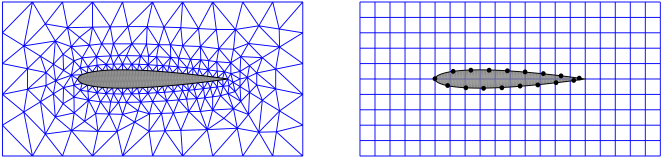
Outline of this lecture

1. Fluid-structure (solid) interaction problem
2. Projection schemes for incompressible Navier-Stokes equations
3. Direct-forcing IB projection methods for FSI problem: one-way coupling and two-way coupling
4. Concluding remarks

Fluid-structure interaction problem (流構耦合問題)

- For computational fluid dynamics (CFD), the primary issues are accuracy, computational efficiency, and the ability to handle complex geometries.
- A fluid-structure interaction (FSI) problem describes the coupled dynamics of fluid mechanics and structure mechanics.
- It usually requires the modeling of complex geometric structure and moving boundaries. It is very challenging for conventional body-fitted approach.



- *I will introduce a Cartesian grid based non-boundary conforming approach, the direct-forcing immersed boundary projection methods.*

Time-dependent incompressible Navier-Stokes equations

Let Ω be an open bounded domain in $\mathbb{R}^d$, $d = 2$ or 3, and let $[0, T]$ be the time interval. The time-dependent, incompressible Navier-Stokes problem can be posed as: find $\boldsymbol{u}$ and p with $\int_\Omega p = 0$, so that

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} - \nu \nabla^2 \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega \times (0, T], \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } \Omega \times (0, T], \\ \boldsymbol{u} &= \boldsymbol{u}_b \quad \text{on } \partial\Omega \times [0, T], \\ \boldsymbol{u} &= \boldsymbol{u}_0 \quad \text{in } \Omega \times \{t = 0\}. \end{aligned}$$

- $\boldsymbol{u}$ is the velocity field, p the pressure (divided by a constant density ρ), ν the kinematic viscosity, $\boldsymbol{f}$ the density of body force.
- *By the divergence theorem, boundary velocity $\boldsymbol{u}_b$ must satisfy*

$$\int_{\partial\Omega} \boldsymbol{u}_b \cdot \boldsymbol{n} \, dA = \int_\Omega \nabla \cdot \boldsymbol{u} \, dV = 0, \quad \forall\, t \in [0, T].$$

Time-discretization of the incompressible NS equations

First, we discretize the time variable of the Navier-Stokes problem, with the spatial variable being left continuous. Consider the implicit Euler time-discretization with explicit first-order approximation to the nonlinear convection term:

$$\begin{aligned} \frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^{n+1} + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n + \nabla p^{n+1} &= \boldsymbol{f}^{n+1} \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{n+1} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^{n+1} &= \boldsymbol{u}_b^{n+1} \quad \text{on } \partial\Omega, \end{aligned}$$

where $t_i := i\Delta t$ for $i = 0, 1, \cdots$, $\Delta t > 0$ is the time step length, and $\boldsymbol{g}^n$ denotes an approximate (or exact) value of $\boldsymbol{g}(t_n)$ at the time level n.

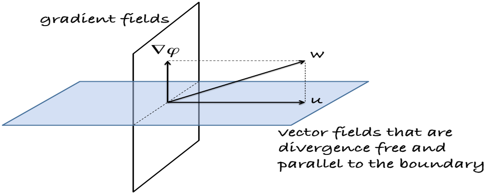
It is highly inefficient in solving this coupled system of Stokes-like equations directly. This is precisely the reason for proposing the projection approach to decouple the computation of $(\boldsymbol{u}^{n+1}, p^{n+1})$.

Helmholtz-Hodge decomposition

Let Ω be an open, bounded, connected, Lipschitz-continuous domain.
A vector field $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$ can be uniquely decomposed orthogonally as

$$\boldsymbol{w} = \boldsymbol{u} + \nabla\varphi, \quad \boldsymbol{u} \in H(\mathrm{div};\Omega) \text{ and } \varphi \in H^1(\Omega),$$

where $\boldsymbol{u}$ has zero divergence $\nabla \cdot \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.



- Orthogonality: $\int_\Omega \boldsymbol{u} \cdot \nabla\varphi \, dV = 0$ (L^2-inner product)
- The HHD describes the decomposition of a flow field $\boldsymbol{w}$ into its divergence-free component $\boldsymbol{u}$ and curl-free component $\nabla\varphi$.
- A. J. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics, 2nd Edition*, Springer-Verlag, New York, 1990.

Chorin projection scheme (Math. Comp. 1968/69)

Step 1: Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^* + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n = \boldsymbol{f}^{n+1} & \text{in } \Omega, \\ \boldsymbol{u}^* = \boldsymbol{u}_b^{n+1} & \text{on } \partial\Omega. \end{cases}$$

Step 2: Determine $\boldsymbol{u}^{n+1}$ and p^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^*}{\Delta t} + \nabla p^{n+1} = \boldsymbol{0} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{n+1} = 0 & \text{in } \Omega, \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \boldsymbol{u}_b^{n+1} \cdot \boldsymbol{n} & \text{on } \partial\Omega, \end{cases}$$

which is equivalent to solving the pressure-Poisson equation with the homogeneous Neumann boundary condition:

$$\begin{cases} \nabla^2 p^{n+1} = \dfrac{1}{\Delta t} \nabla \cdot \boldsymbol{u}^* & \text{in } \Omega, \\ \nabla p^{n+1} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

and then define the velocity field by $\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \Delta t \nabla p^{n+1}$.

Remarks on Chorin's first-order scheme

- *The second step is usually referred to as the projection step.*

$$\boldsymbol{u}^* = \boldsymbol{u}^{n+1} + \Delta t \nabla p^{n+1} = \boldsymbol{u}^{n+1} + \nabla(\Delta t p^{n+1}).$$

  This is indeed the standard HHD of $\boldsymbol{u}^*$ when $\boldsymbol{u}_b^{n+1} = \mathbf{0}$ on $\partial\Omega$.

- Summing all equations in Chorin's projection scheme, we have

$$\begin{aligned} \frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^* + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n + \nabla p^{n+1} &= \boldsymbol{f}^{n+1} \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{n+1} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} &= \boldsymbol{u}_b^{n+1} \cdot \boldsymbol{n} \quad \text{on } \partial\Omega, \end{aligned}$$

  different from the original semi-implicit discretization. Since

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \Delta t \nabla p^{n+1} \approx \boldsymbol{u}^* \text{ in } \Omega \quad \text{as } \Delta t \to 0^+,$$

  it is not surprising that we should expect

$$\nabla^2 \boldsymbol{u}^{n+1} \approx \nabla^2 \boldsymbol{u}^* \text{ in } \Omega \quad \text{and} \quad \boldsymbol{u}^{n+1} \approx \boldsymbol{u}_b^{n+1} \text{ on } \partial\Omega \quad \text{as } \Delta t \to 0^+.$$

Choi-Moin projection scheme (JCP 1994)

Step 1: Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\widetilde{\boldsymbol{u}} - \boldsymbol{u}^n}{\Delta t} - \dfrac{\nu}{2}\nabla^2(\widetilde{\boldsymbol{u}} + \boldsymbol{u}^n) + [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]^{n+\frac{1}{2}} + \nabla p^{n-\frac{1}{2}} = [\boldsymbol{f}]^{n+\frac{1}{2}} \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad \widetilde{\boldsymbol{u}} = \boldsymbol{u}_b^{n+1} \quad \text{on } \partial\Omega; \\ \dfrac{\boldsymbol{u}^* - \widetilde{\boldsymbol{u}}}{\Delta t} - \nabla p^{n-\frac{1}{2}} = \boldsymbol{0} \quad \text{in } \Omega. \end{cases}$$

Step 2: Determine $\boldsymbol{u}^{n+1}$ and φ^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^*}{\Delta t} + \nabla\varphi^{n+1} = \boldsymbol{0} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}^{n+1} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n} = \boldsymbol{u}^*\cdot\boldsymbol{n} \quad \text{on } \partial\Omega. \end{cases}$$

It is equivalent to solving the φ^{n+1}-Neumann Poisson problem:

$$\begin{cases} \nabla^2\varphi^{n+1} = \dfrac{1}{\Delta t}\nabla\cdot\boldsymbol{u}^* \quad \text{in } \Omega, \\ \nabla\varphi^{n+1}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

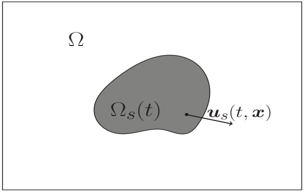
and then set $\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \Delta t\nabla\varphi^{n+1}$.

Step 3: Update the pressure as $p^{n+\frac{1}{2}} = \varphi^{n+1} - \dfrac{\nu}{2}\nabla\cdot\widetilde{\boldsymbol{u}}$.

A fluid-solid interaction problem

A typical one-way coupling problem is flow over a stationary or moving solid ball with a prescribed velocity. Let Ω be the fluid domain which encloses a rigid body positioned at $\overline{\Omega}_s(t)$ *with a prescribed velocity* $\boldsymbol{u}_s(t,\boldsymbol{x})$. The FSI problem with initial value and no-slip boundary condition can be posed as follows:

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} - \nu\nabla^2\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } (\Omega\setminus\overline{\Omega}_s)\times(0,T], \\
\nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } (\Omega\setminus\overline{\Omega}_s)\times(0,T], \\
\boldsymbol{u} &= \boldsymbol{u}_b \quad \text{on } \partial\Omega\times[0,T], \\
\boldsymbol{u} &= \boldsymbol{u}_s \quad \text{on } \partial\Omega_s\times[0,T], \\
\boldsymbol{u} &= \boldsymbol{u}_0 \quad \text{in } (\Omega\setminus\overline{\Omega}_s)\times\{t=0\}.
\end{aligned}
$$



The body-fitted approach

The body-fitted approach is a conventional method for solving the FSI problem. For example, using the implicit Euler discretization at time t_{n+1}, we solve the linearization in the spatial domain $\Omega \setminus \overline{\Omega}_s^{n+1}$

$$
\begin{aligned}
\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^{n+1} + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n + \nabla p^{n+1} &= \boldsymbol{f}^{n+1} \quad \text{in } \Omega \setminus \overline{\Omega}_s^{n+1}, \\
\nabla \cdot \boldsymbol{u}^{n+1} &= 0 \quad \text{in } \Omega \setminus \overline{\Omega}_s^{n+1}, \\
\boldsymbol{u}^{n+1} &= \boldsymbol{u}_b^{n+1} \quad \text{on } \partial\Omega, \\
\boldsymbol{u}^{n+1} &= \boldsymbol{u}_s^{n+1} \quad \text{on } \partial\Omega_s^{n+1}.
\end{aligned}
$$

Again, it is highly inefficient in solving these equations directly. Below, we consider the direct-forcing immersed boundary projection approach.

A direct-forcing approach: virtual force $\boldsymbol{F}$

A virtual force term $\boldsymbol{F}$ is added to the momentum equation to accommodate interaction between the solid and the fluid, and we expect the problem can be solved on the whole domain Ω and do not need to set the interior boundary condition $\boldsymbol{u}_s$ on the interface $\partial\Omega_s$:

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} - \nu\nabla^2\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} + \boldsymbol{F} \quad \text{in } \Omega \times (0,T], \\
\nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega \times (0,T], \\
\boldsymbol{u} &= \boldsymbol{u}_b \quad \text{on } \partial\Omega \times [0,T], \\
\boldsymbol{u} &= \boldsymbol{u}_0 \quad \text{in } \Omega \times \{t=0\}.
\end{aligned}$$

The virtual force $\mathbf{F}$ *exists in the rigid body* $\overline{\Omega}_s(t)$ *which is treated as a portion of the fluid but the virtual force enforces it to act like a solid body. The virtual force will be specified in the time-discrete equations when we apply the projection schemes to solve the time-discretization problem.*

We first consider the first-order projection scheme of Chorin.

A primitive direct-forcing IB projection method (Chorin)

The main idea was proposed by Kajishima *et al.* (JSME-B 2001) & Noor-Chern-Horng (CM 2009).

Step 1: Solve the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^* + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n = \boldsymbol{f}^{n+1} & \text{in } \Omega, \\ \boldsymbol{u}^* = \boldsymbol{u}_b^{n+1} & \text{on } \partial\Omega. \end{cases}$$

Step 2: Determine $\boldsymbol{u}^{**}$ and p^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{**} - \boldsymbol{u}^*}{\Delta t} + \nabla p^{n+1} = \mathbf{0} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{**} = 0 & \text{in } \Omega, \\ \boldsymbol{u}^{**} \cdot \boldsymbol{n} = \boldsymbol{u}_b^{n+1} \cdot \boldsymbol{n} & \text{on } \partial\Omega. \end{cases}$$

It is equivalent to solving the p^{n+1}-Neumann Poisson problem:

$$\begin{cases} \nabla^2 p^{n+1} = \dfrac{1}{\Delta t} \nabla \cdot \boldsymbol{u}^* & \text{in } \Omega, \\ \nabla p^{n+1} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

and set $\boldsymbol{u}^{**} = \boldsymbol{u}^* - \Delta t \nabla p^{n+1} \Longrightarrow \nabla \cdot \boldsymbol{u}^{**} = 0,\ \boldsymbol{u}^{**} \cdot \boldsymbol{n} = \boldsymbol{u}_b^{n+1} \cdot \boldsymbol{n}$

A primitive direct-forcing IB projection method (Chorin)

Step 3: Define the virtual force $\boldsymbol{F}^{n+1}$ and then determine the velocity field $\boldsymbol{u}^{n+1}$ by setting

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^{**}}{\Delta t} = \boldsymbol{F}^{n+1} := \eta \frac{\boldsymbol{u}_s - \boldsymbol{u}^{**}}{\Delta t} \quad \text{in } \Omega,$$

where $\eta(\boldsymbol{x}, t_{n+1})$ is defined by

$$\eta(\boldsymbol{x}, t_{n+1}) = \begin{cases} 1 & \boldsymbol{x} \in \overline{\Omega}_s^{n+1}, \\ 0 & \boldsymbol{x} \notin \overline{\Omega}_s^{n+1}. \end{cases}$$

The virtual force $\boldsymbol{F}^{n+1}$ exists on the whole solid body and zero elsewhere. In other words, in this step, we simply set

$$\boldsymbol{u}^{n+1} = \begin{cases} \boldsymbol{u}^{**} & \text{in } \overline{\Omega} \setminus \overline{\Omega}_s^{n+1}, \\ \boldsymbol{u}_s & \text{in } \overline{\Omega}_s^{n+1}. \end{cases}$$

We remark that η can be taken fractional on the boundary cells when we consider the space-discretization.

Inconsistency in the direct-forcing IB projection method

1. Although the direct-forcing IB projection method seems to produce reasonable results for many fluid-solid interaction problems, *it violates our physical intuition!*
2. It is not always convergent when the direct-forcing IB approach combined with an arbitrary chosen projection scheme, e.g., the scheme of Brown *et al.*, *unless the time step is very small.*

   The reason for this is because the velocity and pressure used in solving the intermediate velocity field $\boldsymbol{u}^*$ *may be not consistent!*

In what follows, we will propose a simple remedy to retrieve the direct-forcing IB projection method.

We will use the idea of the prediction-correction approach to fit the physical intuition and carefully choose a “good” projection scheme!

A direct-forcing IB projection method with PC (Choi-Moin)

Prediction –

Step 1.1: Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\widetilde{\boldsymbol{u}} - \boldsymbol{u}^n}{\Delta t} - \dfrac{\nu}{2}\nabla^2(\widetilde{\boldsymbol{u}} + \boldsymbol{u}^n) + [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]^{n+\frac{1}{2}} + \nabla p^{n-\frac{1}{2}} = [\boldsymbol{f}]^{n+\frac{1}{2}} \quad \text{in } \Omega, \\ \qquad\qquad \widetilde{\boldsymbol{u}} = \boldsymbol{u}_b^{n+1} \quad \text{on } \partial\Omega; \\ \dfrac{\boldsymbol{u}^* - \widetilde{\boldsymbol{u}}}{\Delta t} - \nabla p^{n-\frac{1}{2}} = \boldsymbol{0} \quad \text{in } \Omega. \end{cases}$$

Step 1.2: Determine $\boldsymbol{u}^{**}$ and φ^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{**} - \boldsymbol{u}^*}{\Delta t} + \nabla\varphi^{n+1} = \boldsymbol{0} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}^{**} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^{**}\cdot\boldsymbol{n} = \boldsymbol{u}^*\cdot\boldsymbol{n} \quad \text{on } \partial\Omega. \end{cases}$$

Step 1.3: Predict the virtual force $\widetilde{\boldsymbol{F}}^{n+\frac{1}{2}}$ by setting

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^{**}}{\Delta t} = \widetilde{\boldsymbol{F}}^{n+\frac{1}{2}} := \eta\,\frac{\boldsymbol{u}_s - \boldsymbol{u}^{**}}{\Delta t} \quad \text{in } \Omega.$$

Correction –

Step 2.1: Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\widetilde{\boldsymbol{u}}-\boldsymbol{u}^n}{\Delta t}-\dfrac{\nu}{2}\nabla^2(\widetilde{\boldsymbol{u}}+\boldsymbol{u}^n)+[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]^{n+\frac{1}{2}}+\nabla p^{n-\frac{1}{2}}=[\boldsymbol{f}]^{n+\frac{1}{2}}+\widetilde{\boldsymbol{F}}^{n+\frac{1}{2}} \text{ in } \Omega, \\ \hfill \widetilde{\boldsymbol{u}}=\boldsymbol{u}_b^{n+1} \text{ on } \partial\Omega; \\ \dfrac{\boldsymbol{u}^*-\widetilde{\boldsymbol{u}}}{\Delta t}-\nabla p^{n-\frac{1}{2}}=\boldsymbol{0} \quad \text{in } \Omega. \end{cases}$$

Step 2.2: Determine $\boldsymbol{u}^{**}$ and correct φ^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{**}-\boldsymbol{u}^*}{\Delta t}+\nabla\varphi^{n+1} = \boldsymbol{0} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}^{**} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^{**}\cdot\boldsymbol{n} = \boldsymbol{u}^*\cdot\boldsymbol{n} \quad \text{on } \partial\Omega. \end{cases}$$

Step 2.3: Correct the velocity $\boldsymbol{u}^{n+1}$ and virtual force $\boldsymbol{F}^{n+\frac{1}{2}}$,

$$\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{**}}{\Delta t}=\eta\frac{\boldsymbol{u}_s-\boldsymbol{u}^{**}}{\Delta t} \text{ in } \Omega, \quad \boldsymbol{F}^{n+\frac{1}{2}}=\widetilde{\boldsymbol{F}}^{n+\frac{1}{2}}+\eta\frac{\boldsymbol{u}_s-\boldsymbol{u}^{**}}{\Delta t} \text{ in } \overline{\Omega}_s^{n+1}.$$

Step 2.4: Update the pressure as $p^{n+\frac{1}{2}}=\varphi^{n+1}-\dfrac{\nu}{2}\nabla\cdot\widetilde{\boldsymbol{u}}$.

Space-discretization on a staggered grid

In the following numerical experiments, we employ the two-stage direct-forcing IB projection method (based on Choi-Moin scheme) and apply the second-order centered differences over a staggered grid for space-discretization:

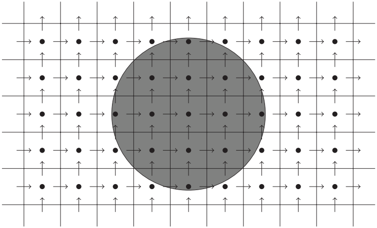


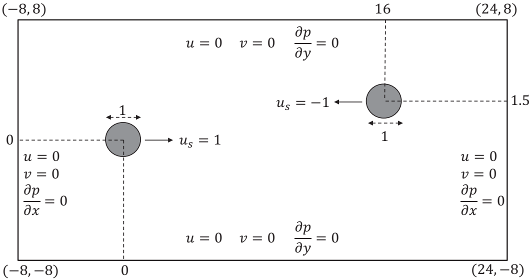
Diagram of the computational domain Ω with staggered grid, where the unknowns u, v and p are approximated at the grid points marked by $\rightarrow$, $\uparrow$ and $\bullet$, respectively

In all examples, the body force $\boldsymbol{f}$ are zero. The volume-of-solid function η is fractional on the boundary cells.

Example 1: two cylinders moving towards each other

Problem setting–

- A uniform grid 640×320 is adopted to discretize the computational domain is $\Omega = (-8, 24) \times (-8, 8)$.
- $\Delta t = 1/200 \quad (CFL$ number is 0.1).
- The Reynolds number is $Re = 40$.



Example 1: two cylinders moving towards each other

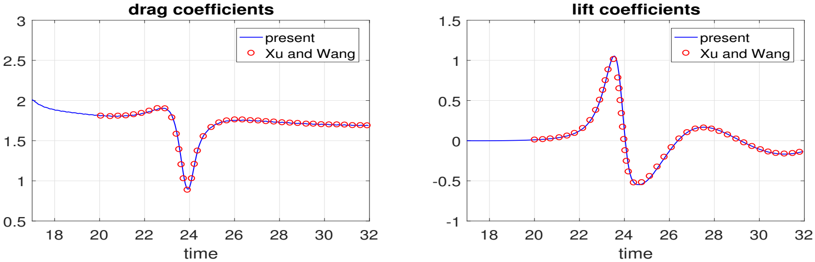
- The motion of the lower and upper cylinders are governed by setting the dynamics of their centers $(x_{\text{lc}}, y_{\text{lc}})$ and $(x_{\text{uc}}, y_{\text{uc}})$ to

$$x_{\text{lc}} = \begin{cases} \frac{4}{\pi} \sin\left(\frac{\pi t}{4}\right), & 0 \leq t \leq 16, \\ t - 16, & 16 \leq t \leq 32 \end{cases} \quad \text{and} \quad y_{\text{lc}} = 0,$$

and

$$x_{\text{uc}} = \begin{cases} 16 - \frac{4}{\pi} \sin\left(\frac{\pi t}{4}\right), & 0 \leq t \leq 16, \\ 32 - t, & 16 \leq t \leq 32 \end{cases} \quad \text{and} \quad y_{\text{uc}} = 1.5.$$

Example 1: two cylinders moving towards each other



The time evolution of drag and lift coefficients, C_d and C_ℓ, for the upper cylinder in the flow around two cylinders compared with the results of Xu-Wang (JCP 2006)

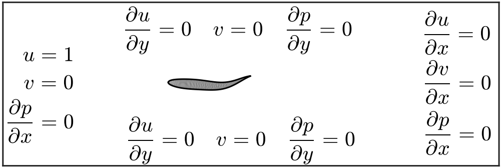
$$F_d = -\int_\Omega F_1 d\boldsymbol{x} = -\int_{\Omega_s} F_1 d\boldsymbol{x} \approx -\sum_{\boldsymbol{x}_{ij}} F_1 h^2 \quad \text{and} \quad C_d = \frac{F_d}{U_\infty^2 D/2},$$

$$F_\ell = -\int_\Omega F_2 d\boldsymbol{x} = -\int_{\Omega_s} F_2 d\boldsymbol{x} \approx -\sum_{\boldsymbol{x}_{ij}} F_2 h^2 \quad \text{and} \quad C_\ell = \frac{F_\ell}{U_\infty^2 D/2},$$

Example 2: fish swimming

Problem setting–

- Reynolds number is defined as $Re = U_\infty L/\nu$, where L is the chord length of wavy foil. In this simulation, $L = U_\infty = 1$ and $Re = 5000$.
- Computational domain size is $6L \times 2L$, $\Omega = (-2, 4) \times (-1, 1)$.
- $\Delta x = \Delta y = 1/480$, $\Delta t = 0.0002$, CFL number is 0.096, and the final time $T = 20$.



Please see some animations of the numerical simulations.

The governing equations of freely falling solid body

Consider a 2-D solid object of constant density ρ_s positioned at $\overline{\Omega}_s$ with centroid $\boldsymbol{X}_c$, translational velocity $\boldsymbol{u}_c$ and angular velocity ω. The velocity of the solid object is given by

$$\boldsymbol{u}_s(t,\boldsymbol{x}) = \boldsymbol{u}_c(t) + \omega(t) \times \boldsymbol{r}(t,\boldsymbol{x}), \quad \boldsymbol{r} := \boldsymbol{x} - \boldsymbol{X}_c, \quad \forall\, \boldsymbol{x} \in \overline{\Omega}_s(t).$$

From Newton's second law, we have

$$\begin{aligned} \frac{d\boldsymbol{u}_c}{dt}\int_{\Omega_s} \rho_f \, dV &= \int_{\partial\Omega_s} \boldsymbol{\sigma}\cdot\boldsymbol{n}\, dS + \int_{\Omega_s} \rho_f \boldsymbol{F}\, dV + \int_{\Omega_s} \rho_f \boldsymbol{g}\, dV, \\ \boldsymbol{I}_f \frac{d\omega}{dt} &= \int_{\partial\Omega_s} \boldsymbol{r}\times(\boldsymbol{\sigma}\cdot\boldsymbol{n})\, dS + \int_{\Omega_s} \rho_f \boldsymbol{r}\times\boldsymbol{F}\, dV, \end{aligned}$$

where $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu_f \boldsymbol{\varepsilon}(\boldsymbol{u})$ is the stress tensor of the fluid, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the rate of strain tensor, μ_f is the dynamic viscosity, $\boldsymbol{n}$ is the outward unit normal vector on $\partial\Omega_s$, ρ_f is the density of fluid, $\boldsymbol{g}$ is the gravity, $\boldsymbol{I}_f (= \int_{\Omega_s} \rho_f |\boldsymbol{r}|^2 dV)$ is the rotational inertia for the fluid, and $\boldsymbol{F}$ is the virtual force, which is chosen to ensure $\boldsymbol{u} = \boldsymbol{u}_s$ on $\overline{\Omega}_s$.

From the viewpoint of solid body

The motion of solid object can also be described by translational and angular momentum of the solid body. Thus, we have

$$\begin{aligned} \frac{d\boldsymbol{u}_c}{dt}\int_{\Omega_s}\rho_s\, dV &= \int_{\partial\Omega_s}\boldsymbol{\sigma}\cdot\boldsymbol{n}dS + \int_{\Omega_s}\rho_s\boldsymbol{g}\, dV, \\ \boldsymbol{I}_s\frac{d\omega}{dt} &= \int_{\partial\Omega_s}\boldsymbol{r}\times(\boldsymbol{\sigma}\cdot\boldsymbol{n})\, dS, \end{aligned}$$

where $\boldsymbol{I}_s(=\int_{\Omega_s}\rho_s|\boldsymbol{r}|^2dV)$ is the rotational inertia for the solid object, ρ_s is the density of solid. Since the virtual force $\boldsymbol{F}$ is chosen to make these two systems are equivalent, so we have the following equations of motion:

$$\begin{aligned} \frac{d\boldsymbol{u}_c}{dt}\overbrace{\int_{\Omega_s}(\rho_s-\rho_f)\, dV}^{M_s-M_f} &= \overbrace{\int_{\Omega_s}(\rho_s-\rho_f)\boldsymbol{g}dV}^{(M_s-M_f)\boldsymbol{g}} - \int_{\Omega_s}\rho_f\boldsymbol{F}\, dV, \\ (\boldsymbol{I}_s-\boldsymbol{I}_f)\frac{d\omega}{dt} &= -\int_{\Omega_s}\rho_f\boldsymbol{r}\times\boldsymbol{F}\, dV. \end{aligned}$$

The two-way fluid-solid interaction problem

The fluid-solid interaction of the freely falling solid body with a virtual force can be formulated as the following initial-boundary value problem: *find* $\boldsymbol{u}$, p, $\boldsymbol{F}$, $\boldsymbol{u}_c$ *and* ω *with* $\int_\Omega p = 0$ *such that*

$$
\begin{gathered}
\frac{\partial \boldsymbol{u}}{\partial t} - \nu \nabla^2 \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} + \boldsymbol{F} \quad t \in (0, T], \quad \boldsymbol{x} \in \Omega, \\
\nabla \cdot \boldsymbol{u} = 0 \quad t \in (0, T], \quad \boldsymbol{x} \in \Omega, \\
\boldsymbol{u} = \boldsymbol{u}_b \quad t \in (0, T], \quad \boldsymbol{x} \in \partial\Omega, \\
\boldsymbol{u} = \boldsymbol{u}_0 \quad t = 0, \quad \boldsymbol{x} \in \overline{\Omega},
\end{gathered}
$$

$$
\begin{gathered}
\boldsymbol{u} = \boldsymbol{u}_s := \boldsymbol{u}_c + \omega \times \boldsymbol{r} \quad \text{in } \overline{\Omega}_s, \\
(M_s - M_f)\frac{d\boldsymbol{u}_c}{dt} = (M_s - M_f)\boldsymbol{g} - \int_{\Omega_s} \rho_f \boldsymbol{F} dV, \quad \boldsymbol{u}_c(0) = \boldsymbol{u}_{c0}, \\
(\boldsymbol{I}_s - \boldsymbol{I}_f)\frac{d\omega}{dt} = -\int_{\Omega_s} \rho_f \boldsymbol{r} \times \boldsymbol{F} dV, \quad \omega(0) = \omega_0.
\end{gathered}
$$

Time-discretization of the equations of motion

At the time level t_{n+1}, we compute the translational velocity and the angular velocity, denoted by $\boldsymbol{u}_c^{n+1}$ and ω^{n+1}, by considering

$$\begin{aligned} M_s \frac{\boldsymbol{u}_c^{n+1} - \boldsymbol{u}_c^n}{\Delta t} &= (M_s - M_f)\boldsymbol{g} - \int_{\Omega_s^n} \rho_f \boldsymbol{F}^n dV + M_f \frac{\boldsymbol{u}_c^n - \boldsymbol{u}_c^{n-1}}{\Delta t}, \\ \boldsymbol{I}_s \frac{\omega^{n+1} - \omega^n}{\Delta t} &= -\int_{\Omega_s^n} \rho_f \boldsymbol{r}^n \times \boldsymbol{F}^n dV + \boldsymbol{I}_f \frac{\omega^n - \omega^{n-1}}{\Delta t}. \end{aligned}$$

Once $\boldsymbol{u}_c^{n+1}$ and ω^{n+1} are obtained, we compute the solid center and rotational angle by taking

$$\frac{\boldsymbol{X}_c^{n+1} - \boldsymbol{X}_c^n}{\Delta t} = \boldsymbol{u}_c^{n+1}, \quad \frac{\theta^{n+1} - \theta^n}{\Delta t} = \omega^{n+1},$$

then update the solid domain Ω_s^{n+1} and set the solid velocity by

$$\boldsymbol{u}_s^{n+1} = \boldsymbol{u}_c^{n+1} + \omega^{n+1} \times \boldsymbol{r}^{n+1} \ \text{with} \ \boldsymbol{r}^{n+1} = \boldsymbol{X} - \boldsymbol{X}_c^{n+1}.$$

A two-stage direct-forcing IB projection method

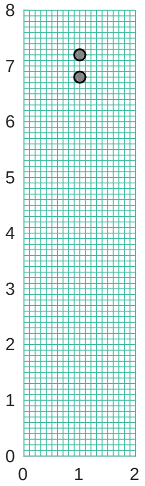
- Based on the time-discretization of the equations of motion, we can design a two-stage direct-forcing IB projection method for FSI problems without prescribed solid velocity.
- In case where multiple bodies exist in fluid, a collision model is generally needed to avoid particles overlapping, see next page.

A simple collision model

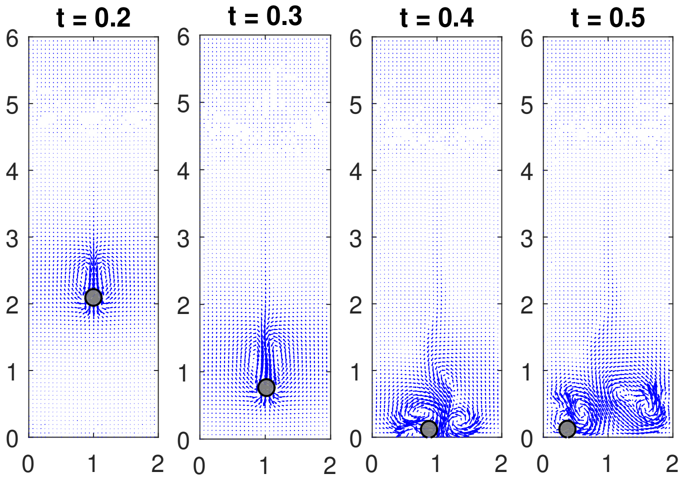
Singh-Joseph-Hesla-Glowinski-Pan (潘從輝) (JCP 2000) introduced an additional body force, called repulsion force, arising in body-body or body-wall collision:

$$\boldsymbol{F}_{\text{co}}^{ij} = \begin{cases} 0, & \text{if } d_{ij} > R_i + R_j + \delta, \\ \dfrac{(\boldsymbol{X}_c^{(i)} - \boldsymbol{X}_c^{(j)})}{\varepsilon}(R_i + R_j + \delta - d_{ij})^2, & \text{otherwise,} \end{cases}$$

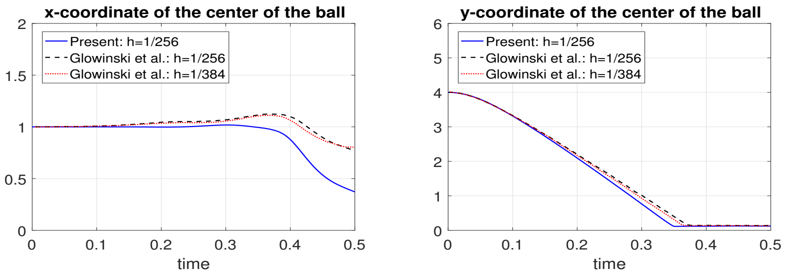
where d_{ij} is the distance between the center of the ith and jth particles and R_i and R_j are their radius, respectively. The δ is the range within which the repulsive force acts on both bodies, ε is the smaill positive coefficient.



Flow field visualization



Time evolution of position



Please see some animations of the numerical simulations of freely falling solid bodies in an incompressible viscous fluid.

A new direct-forcing IB projection method

Step 1: Find Ω_s^{n+1} and $\boldsymbol{F}^{n+1}$ by the following algorithm

Set $\boldsymbol{X}_c^{n+1,0} = \boldsymbol{X}_c^n$, $\boldsymbol{u}_c^{n+1,0} = \boldsymbol{u}_c^n$, and $\boldsymbol{F}^{n+1,0} = \boldsymbol{F}^n$. For $k = 1, \cdots, N$

$$
\begin{aligned}
\widehat{\boldsymbol{u}}_c^{n+1,k} &= \boldsymbol{u}_c^{n+1,k-1} + \frac{\Delta t}{N} g - \frac{(\Delta t/N)}{M_s - M_f} \int_{\Omega_s(\boldsymbol{X}_c^{n+1,k-1})} \rho_f \boldsymbol{F}^{n+1,k-1}\, dx, \\
\widehat{\boldsymbol{X}}_c^{n+1,k} &= \boldsymbol{X}_c^{n+1,k-1} + \frac{\Delta t}{2N} \left(\widehat{\boldsymbol{u}}_c^{n+1,k} + \boldsymbol{u}_c^{n+1,k-1} \right), \\
\boldsymbol{u}_c^{n+1,k} &= \boldsymbol{u}_c^{n+1,k-1} + \frac{\Delta t}{N} g - \frac{\Delta t/N}{2(M_s - M_f)} \left\{ \int_{\Omega_s(\widehat{\boldsymbol{X}}_c^{n+1,k})} \rho_f \frac{\boldsymbol{u}_c^{n+1,k} - \boldsymbol{u}^n}{k\Delta t/N} - \mathrm{RHS}^n\, dx \right. \\
&\qquad \left. + \int_{\Omega_s(\boldsymbol{X}_c^{n+1,k-1})} \rho_f \boldsymbol{F}^{n+1,k-1}\, dx \right\} \\
\boldsymbol{X}_c^{n+1,k} &= \boldsymbol{X}_c^{n+1,k-1} + \frac{\Delta t}{2N} \left(\boldsymbol{u}_c^{n+1,k} + \boldsymbol{u}_c^{n+1,k-1} \right) \\
\boldsymbol{F}^{n+1,k} &= \frac{\boldsymbol{u}_c^{n+1,k} - \boldsymbol{u}^n}{k\Delta t/N} - \mathrm{RHS}^n
\end{aligned}
$$

Define $\boldsymbol{X}_c^{n+1} = \boldsymbol{X}_c^{n+1,N}$, $\boldsymbol{u}_c^{n+1} = \boldsymbol{u}_c^{n+1,N}$ and $\boldsymbol{F}^{n+1} = \boldsymbol{F}^{n+1,N}$.

A new direct-forcing IB projection method (cont'd)

Step 1: Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{3\widetilde{\boldsymbol{u}} - 4\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{2\Delta t} - \nu \nabla^2 \widetilde{\boldsymbol{u}} + [(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}]^{n+1} + \nabla p^n = \boldsymbol{f}^{n+1} + \boldsymbol{F}^{n+1} & \text{in } \Omega, \\ \widetilde{\boldsymbol{u}} = \boldsymbol{u}_b^{n+1} & \text{on } \partial\Omega; \end{cases}$$

$$\frac{3\boldsymbol{u}^* - 3\widetilde{\boldsymbol{u}}}{2\Delta t} - \nabla p^n = \boldsymbol{0} \quad \text{in } \Omega.$$

Step 2: Determine $\boldsymbol{u}^{n+1}$ and φ^{n+1} by solving

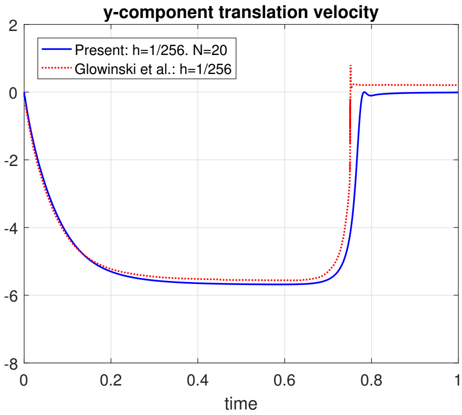
$$\begin{cases} \dfrac{3\boldsymbol{u}^{n+1} - 3\boldsymbol{u}^*}{2\Delta t} + \nabla \varphi^{n+1} = \boldsymbol{0} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{n+1} = 0 & \text{in } \Omega, \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \boldsymbol{u}^* \cdot \boldsymbol{n} & \text{on } \partial\Omega. \end{cases}$$

Step 3: Update the pressure as $p^{n+1} = \varphi^{n+1} - \nu \nabla \cdot \widetilde{\boldsymbol{u}}$.

Problem setting of sedimentation of a circular body: $\nu = 0.1$

- The computational domain $\Omega = (0,2) \times (0,6)$.
- The diameter of the body is $d = 0.25$ and is located at $(1,4)$ at time $t = 0$.
- The fluid density is $\rho_f = 1$ and the disk density $\rho_s = 1.25$.
- $\nu = 0.1$, $h = 1/256$, and $\Delta t = 7.5 \times 10^{-4}$.

Numerical results of sedimentation: $\nu = 0.1$

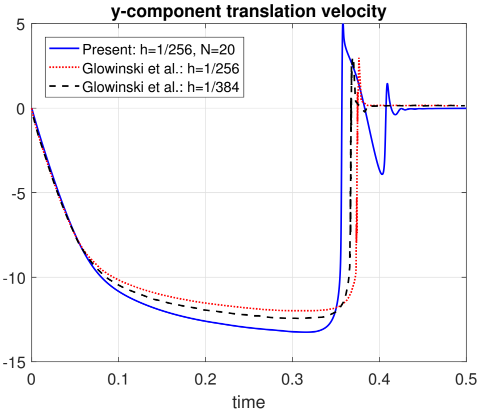


R. Glowinski, T. W. Pan, T. I. Hesla, D. D. Joseph, and J. Périaux,
A fictitious domain approach to the direct numerical simulation of incompressible viscous flow past moving rigid bodies: application to particulate flow, *JCP*, 169 (2001), pp. 363-426.

Problem setting of sedimentation of a circular body: $\nu = 0.01$

- The computational domain $\Omega = (0, 2) \times (0, 6)$.
- The diameter of the body is $d = 0.25$ and is located at $(1, 4)$ at time $t = 0$.
- The fluid density is $\rho_f = 1$ and the disk density $\rho_s = 1.5$.
- $\nu = 0.01$, $h = 1/256$, and $\Delta t = 7.5 \times 10^{-5}$.

Numerical results of sedimentation: $\nu = 0.01$



Concluding remarks

1. We have developed a successful two-stage direct-forcing IB projection method for simulating the fluid-solid interaction problems, where the immersed solid object can be moving with a prescribed velocity.
2. Details of the above approach can be found in

   T.-L. Horng, P.-W. Hsieh, S.-Y. Yang*, and C.-S. You,

   A simple direct-forcing immersed boundary projection method with prediction-correction for fluid-solid interaction problems, *Computers & Fluids*, in press, 2018.
3. Further works are needed, including efficient extensions of the method to solve the freely falling body in an incompressible viscous fluid and the fluid-elastic body interaction problems.

Thank you for your attention!