Mathematical Approaches to 2-D Image Processing: A Preliminary Exploration

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Image processing and computational mathematics

- Tony F. Chan: *Image processing and computational mathematics,* Davis Centre, University of Waterloo, October 7th, 2015.

  https://www.youtube.com/watch?v=zZN_L8ntO9I

  Image processing has emerged not only as an application domain where computational mathematics provides ideas and solutions but also in spurring new research directions,

  "new Computational Fluid Dynamics!"

- We briefly introduce two different mathematical approaches to 2-D image processing:

  (1) *Variational method/energy functional minimization*

  (2) *Sparse representation and dictionary learning*

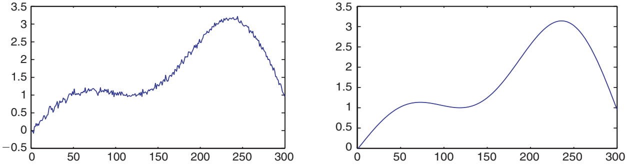
Total variation

Let $u : [a,b] \to \mathbb{R}$. Let $\mathcal{P}_n = \{x_0 = a, x_1, \cdots, x_n = b\}$ be an arbitrary partition of $\overline{\Omega} := [a,b]$ and $\Delta x_i = x_i - x_{i-1}$. The total variation of u is

$$\begin{aligned} \|u\|_{TV(\Omega)} &:= \sup_{\mathcal{P}_n} \sum_{i=1}^{n} |u(x_i) - u(x_{i-1})| = \sup_{\mathcal{P}_n} \sum_{i=1}^{n} \left| \frac{u(x_i) - u(x_{i-1})}{\Delta x_i} \right| \Delta x_i \\ &= \int_\Omega |u'(x)|\, dx, \quad \text{if } u \text{ is smooth.} \end{aligned}$$

Denoising is the problem of removing noise from an image:

$$\text{minimize} \left(\int_\Omega |u'(x)|\, dx + \text{ some data fidelity term} \right).$$



1-D example

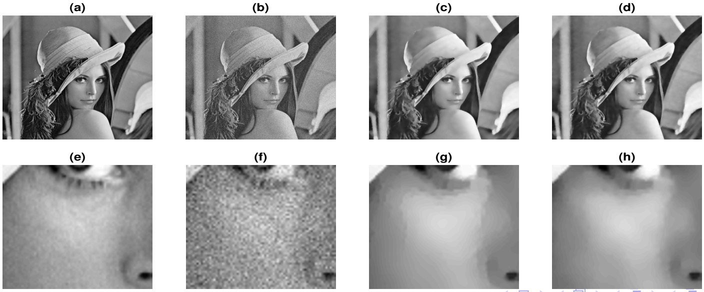
ROF total-variation model vs. adaptive diffusivity model

Let $f : \overline{\Omega} \subset \mathbb{R}^2 \to \mathbb{R}$ be a given noisy image. Rudin-Osher-Fatemi (1992) proposed the model:

$$\min_{u \in \mathcal{V}} \left(\int_{\Omega} |\nabla u| + \frac{\lambda}{2}(u - f)^2 \, dx \right), \quad \lambda > 0.$$

Hsieh-Shao-Yang (2018) proposed an adaptive model to alleviate *the staircasing effect:*

$$\min_{u \in \mathcal{V}} \left(\int_{\Omega} \frac{1}{2} \varphi(|\nabla u^*|) |\nabla u|^2 + \frac{\lambda}{2}(u - f)^2 \, dx \right), \quad \lambda > 0.$$



A variational model for image contrast enhancement

Hsieh-Shao-Yang (2020): for every $f \in \{f_R, f_G, f_B\}$, we solve

$$\min_{u \in \mathcal{V}} \left(\int_\Omega |\nabla u - \nabla h_c| \, d\boldsymbol{x} + \frac{\lambda}{2} \int_\Omega (u - g_c)^2 \, d\boldsymbol{x} \right),$$

where the adaptive functions g_c and h_c are defined as

$$g_c(\boldsymbol{x}) := \begin{cases} \alpha \bar{f}, & \boldsymbol{x} \in \Omega_d, \\ f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_b, \end{cases} \qquad h_c(\boldsymbol{x}) := \begin{cases} \beta f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_d, \\ f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_b. \end{cases}$$

Numerical methods: *(i) Euler-Lagrange equation + solving IBVP; (ii) direct discretization + split Bregman iterations.*

Numerical results by the split Bregman iterations

Chan-Vese segmenation model: nonconvex minimization

Chan-Vese (1999) modified Mumford-Shah model (1989): two-phase

$$\min_{c_1,c_2,\mathcal{C}} \left(\mu|\mathcal{C}| + \nu|\Omega_{\text{in}}| + \lambda_1 \int_{\Omega_{\text{in}}} \left(f(\boldsymbol{x}) - c_1\right)^2 + \lambda_2 \int_{\Omega_{\text{out}}} \left(f(\boldsymbol{x}) - c_2\right)^2 \right).$$

In terms of H, δ, and the level set function ϕ, we have

$$\min_{c1,c2,\phi} \Big(\mu \int_\Omega \delta(\phi(\boldsymbol{x}))|\nabla\phi(\boldsymbol{x})| + \nu \int_\Omega H(\phi(\boldsymbol{x})) + \lambda_1 \int_\Omega \left(f(\boldsymbol{x}) - c_1\right)^2 H(\phi(\boldsymbol{x}))$$
$$+\lambda_2 \int_\Omega \left(f(\boldsymbol{x}) - c_2\right)^2 \left(1 - H(\phi(\boldsymbol{x}))\right) \Big).$$

Numerical method: *an alternating iterative scheme (region averages + solving IBVP of the Euler-Lagrange equation)*

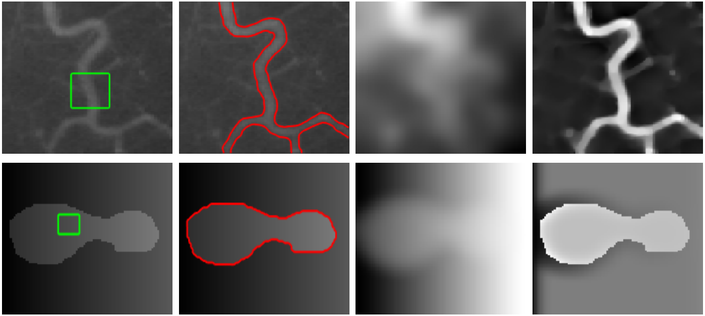
Numerical results by an alternating iterative scheme

Adaptive model for intensity inhomogeneous images

Liao-Yang-You (2022) proposed an entropy-weighted local intensity clustering-based model starting from *the bias field model:* $f = bI + n$*:*

$$\min_{\mathcal{C},b,\boldsymbol{c}} \left(\mu\, |\mathcal{C}| + \int_{\Omega} E_r(\boldsymbol{y}) \sum_{i=1}^{n} \int_{\Omega_i} K(\boldsymbol{y}-\boldsymbol{x}) \big(f(\boldsymbol{x}) - b(\boldsymbol{y})c_i\big)^2 \, d\boldsymbol{x}\, d\boldsymbol{y} \right).$$

Numerical method: *a new alternating iterative scheme, called iterative convolution-thresholding (ICT) scheme.*



initial contour, segmented result, bias field b*, and corrected image* f/b

Sparse representation and dictionary learning

SR problem: *Given a signal vector* $\boldsymbol{x} \in \mathbb{R}^m$ *and a dictionary matrix* $\boldsymbol{D} \in \mathbb{R}^{m \times n}$, $n \gg m$, *we seek a coefficient vector* $\boldsymbol{z}^* \in \mathbb{R}^n$ *such that*

$$\boldsymbol{z}^* = \arg\min_{\boldsymbol{z}} \left(\frac{1}{2} \|\boldsymbol{x} - \boldsymbol{D}\boldsymbol{z}\|_2^2 + \lambda \|\boldsymbol{z}\|_1 \right), \qquad \lambda > 0.$$

SDL problem: *Let* $\{\boldsymbol{x}_i\}_{i=1}^N \subset \mathbb{R}^m$ *be a given dataset of signals. We seek a dictionary matrix* $\boldsymbol{D} = [\boldsymbol{d}_1, \boldsymbol{d}_2, \cdots, \boldsymbol{d}_n] \in \mathbb{R}^{m \times n}$ *together with the sparse coefficient vectors* $\{\boldsymbol{z}_i\}_{i=1}^N \subset \mathbb{R}^n$ *that solve the minimization problem:*

$$\min_{\boldsymbol{D}, \{\boldsymbol{z}_i\}} \left(\frac{1}{2} \sum_{i=1}^N \|\boldsymbol{x}_i - \boldsymbol{D}\boldsymbol{z}_i\|_2^2 + \lambda \sum_{i=1}^N \|\boldsymbol{z}_i\|_1 \right)$$
$$\text{subject to } \|\boldsymbol{d}_k\|_2 \le 1, \ \forall\, 1 \le k \le n, \qquad \lambda > 0.$$

Numerical method: *alternating direction method of multipliers (ADMM).*

Some applications in image processing

Single image inpainting: *we use the complete patches to train the dictionary, recover the incomplete patches by the sparse representation.*

Other applications: *single image super-resolution, image fusion, ...*

Concluding remarks

Still, there are some other mathematical approaches and important techniques in image processing that need to be further studied, e.g.,

- *Robust PCA (SVD), sparse and low-rank representation, ...*
- *Fast and efficient global registration, convex relaxation techniques, ...*

Image stitching

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Thank you for your attention!